NARAJOLE RAJ COLLEGE

Department of Mathematics

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:- Homogeneous Linear Equations :-

A linear differential equation of the form

$$x^n \frac{d^n y}{dx^n} + p_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + p_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + p_{n-1} x \frac{dy}{dx} + p_n y = X \quad \text{——} (1)$$

where $p_1, p_2, \ldots, p_n$ are constants and X is either a function of x or a constant, is known as homogeneous linear equation.

This equation is also known as a Euler-Cauchy equation.

Method of solution of Homogeneous Linear Equations

Method-1 : we put, $x = e^z$ or, $z = \log x$ ——— (2)

Now, $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz}$. $\left[\because \frac{dz}{dx} = \frac{1}{x}\right]$

or, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2} \cdot \frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2} \cdot \frac{dz}{dx}$

$= -\frac{1}{x^2} \cdot \frac{dy}{dz} + \frac{1}{x^2} \cdot \frac{d^2y}{dz^2} = \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$

Similarly,

$$\frac{d^3y}{dx^3} = \frac{1}{x^3}\left(\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}\right)$$

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$$\frac{d^n y}{dx^n} = \frac{1}{x^n}\left[\frac{d^n y}{dz^n} - \frac{n(n-1)}{2}\frac{d^{n-1}y}{dz^{n-1}} + \cdots + (-1)^{n-1}\cdot(n-1)!\frac{dy}{dz}\right]$$

put, $Dy = \frac{dy}{dx}$, $D^2y = \frac{d^2y}{dx^2}$ etc

and $\delta y = \frac{dy}{dz}$, $\delta^2 y = \frac{d^2y}{dz^2}$ etc.

then we get, $xDy = \delta y$

$x^2 D^2 y = \delta^2 y - \delta y = \delta(\delta - 1) y$

$x^3 D^3 y = \delta(\delta-1)(\delta-2) y$ .. etc.

From the equation ①, becomes

$$[\delta(\delta-1)(\delta-2)\cdots(\delta-n+1) + f_1\delta(\delta-1)(\delta-2)\cdots(\delta-n+2) + \cdots + f_{n-1}\delta + f_n]y = Z$$

where, Z is a function of x and we write in this way $\phi(\delta)y = Z$

where, $\phi(\delta) = \delta(\delta-1)\cdots(\delta-n+1) + f_1\delta(\delta-1)\cdots(\delta-n+2) + \cdots + f_{n-1}\delta + f_n$ ——— ④

This is a linear equation with constant co-efficients and solved by previous discussed method.

Now, put $z = \log x$ in the solution ③, we get the solution of the equation ①.

Method-2 :-

In this method we calculate c.f. directly without give any transformation.

Now, $x^n \frac{d^n x^m}{dx^n} = x^n \cdot m(m-1)\cdots(m-n+1)x^{m-n}$

$= m(m-1)\cdots(m-n+1)\cdot x^m$

putting, $y = x^m$ in L.H.S. of equation ①,

L.H.S. $= [m(m-1)(m-2)\cdots(m-n+1) + f_1 m(m-1)\cdots(m-n+2) + \cdots + f_{n-1}m + f_n]x^m$.

$= \phi(m)x^m$,

where, $\phi(m)$ is same function as ④.

Now, if $\phi(m) = 0$, then $y = x^m$ satisfies the equation

$$x^n \frac{d^n y}{dx^n} + f_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + f_{n-1}x\frac{dy}{dx} + f_n y = 0.$$

whose solution gives the c.f. of the solution of ① and it's c.f. is composed of the terms take x^m.

If $\phi(m) = 0$ has the roots $m_1, m_2, \cdots, m_n$, the c.f. will be like

$$\text{c.f.} = c_1 x^{m_1} + c_2 x^{m_2} + \cdots + c_n x^{m_n}$$

Now put $x = e^z \Rightarrow z = \log x$, C.F. of the equation (5) is $\phi(\delta)y = Z$

will given as C.F. $= c_1 e^{m_1 z} + c_2 e^{m_2 z} + \cdots + c_n e^{m_n z}$

and P.I. $= \frac{1}{\phi(\delta)} Z$

thus $y =$ C.F. + P.I. is the complete solution of the equation $\phi(\delta)y = Z$

Equations that can be reduced to Homogeneous Linear form :-

The equations of the form.

$$(a+bx)^n \frac{d^n y}{dx^n} + p_1 (a+bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_{n-1}(a+bx)\frac{dy}{dx} + p_n y = X \quad \text{---(1)}$$

where, $p_1, p_2, \ldots, p_n$ are constants and X is a function of x only can easily be reduced to a homogeneous linear form by putting $z = a + bx$.

when, $z = a + bx$,

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = b \cdot \frac{dy}{dz}$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(b \frac{dy}{dz}\right) = \frac{d}{dz}\left(b \frac{dy}{dz}\right) \cdot \frac{dz}{dx} = b^2 \cdot \frac{d^2 y}{dz^2}$$

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$$\frac{d^n y}{dx^n} = b^n \cdot \frac{d^n y}{dz^n}$$

Putting the values in (1), we get the equation as,

$$z^n \frac{d^n y}{dz^n} + \frac{p_1}{b} \cdot z^{n-1} \cdot \frac{d^{n-1} y}{dz^{n-1}} + \cdots + \frac{p_{n-1}}{b^{n-1}} \frac{dy}{dz} + \frac{p_n}{b^n} y = Z \quad \text{---(2)}$$

where Z is the function of z obtained from X by putting $x = \frac{z-a}{b}$,

The new equation (2) is a homogeneous linear equation and can be solved by using previous methods

Example :-

1. $x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10\left(x+\frac{1}{x}\right)$

put, $x = e^z \Rightarrow z = \log x$ $\therefore x\frac{d}{dx} \equiv \frac{d}{dz} \equiv D$ (say)

then, the given equation is

$\{D(D-1)(D-2)\}y + 2\{D(D-1)\}y + 2y = 10(e^z + e^{-z})$

or, $(D^3 - D^2 + 2)y = 10(e^z + e^{-z})$

The auxiliary equation is $m^3 - m^2 + 2 = 0$

or, $(m+1)(m^2 - 2m + 2) = 0$

$\therefore m = -1, 1 \pm 2i$

Hence, C.F. $= c_1e^{-z} + e^z(c_2\cos z + c_3\sin z)$,

where c_1, c_2, c_3 are arbitrary constants

P.I. $= \frac{1}{D^3 - D^2 + 2}\,10(e^z + e^{-z})$

$= \frac{1}{(D+1)(D^2 - 2D + 2)}\,10(e^z + e^{-z})$

$= \frac{10e^z}{(1+1)(1^2 - 2\cdot 1 + 2)} + 10e^{-z}\cdot\frac{1}{(D-1+1)\{(D-1)^2 - 2(D-1) + 2\}}\,1$

$= \frac{10e^z}{2} + 10e^{-z}\cdot\frac{1}{D(D^2 - 4D + 5)}(1)$

$= 5e^z + \frac{10}{5}e^{-z}\cdot\frac{1}{D}\left(1 + \frac{D^2 - 4D}{5}\right)^{-1}(1)$

$= 5e^z + 2e^{-z}\cdot\frac{1}{D}\left(1 - \frac{D^2 - 4D}{5} + \cdots\right)(1)$

$= 5e^z + 2e^{-z}\cdot\frac{1}{D}(1)$

$= 5e^z + 2ze^{-z}$

The general solution is, $y = c_1e^{-z} + e^z(c_2\cos z + c_3\sin z) + \text{P.I.}$

$y = (c_1 + 2z)e^{-z} + (c_2\cos z + c_3\sin z + 5)e^z$

$= \frac{c_1 + 2\log x}{x} + x\{c_2\cos(\log x) + c_3\sin(\log x) + 5\}$.

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Ex.2 : $(x^2D^2+3xD+1)y = \frac{1}{(1-x)^2}$ [T.U. 2018]

➡ Put, $x = e^z \Rightarrow z = \log x$ $\therefore x\frac{d}{dx} = \frac{d}{dz} = D$ (say).

then the given equation reduces to,

$$\{D(D-1)+3D+1\}y = \frac{1}{(1-e^z)^2}$$

or, $(D+1)^2 y = \frac{1}{(1-e^z)^2}$

Let, $y = e^{mz}$ $(\neq 0)$ be the trial solution of the auxiliary equation

$(m+1)^2 = 0 \Rightarrow m = -1, -1$.

Hence, C.F. $= (c_1 + c_2 z)\, e^{-z}$, where c_1, c_2 are arbitrary constants.

$\therefore$ P.I. $= \frac{1}{(D+1)^2}\left(\frac{1}{(1-e^z)^2}\right)$ ——— (1)

Now, putting $v = \frac{1}{(D+1)}\left(\frac{1}{(1-e^z)^2}\right)$ ——— (2)

then, $(D+1)v = \frac{1}{(1-e^z)^2}$ ——— (3)

or, $\frac{dv}{dz} + v = \frac{1}{(1-e^z)^2}$, which is a linear equation.

then, I.F. $= e^{\int dz} = e^z$

Multiplying both sides of (3) by I.F. and integrating we get,

$$ve^z = \int \frac{e^z\, dz}{(1-e^z)^2} = \frac{1}{1-e^z}$$

$$\therefore v = \frac{1}{e^z(1-e^z)} \quad \text{——— (4)}$$

Now, P.I. $= \frac{1}{(D+1)}\left[\frac{1}{(D+1)}\left(\frac{1}{(1-e^z)^2}\right)\right]$

$$= \frac{1}{(D+1)} v$$

$$= \frac{1}{(D+1)} \cdot \left(\frac{1}{e^z(1-e^z)}\right) \quad [\text{From (4)}]$$

Now, put, $w = \frac{1}{(D+1)} \cdot \frac{1}{e^z(1-e^z)}$

or, $(D+1)w = \frac{1}{e^z(1-e^z)}$

or, $\frac{dw}{dz} + w = \frac{1}{1-e^z} + \frac{1}{e^z}$ ——— (5)

which is also a linear equation.

so, I.F. $= e^{\int dz} = e^z$.

Multiplying both sides of (5) by I.F. and integrating we get,

$$we^z = \int \left(\frac{e^z}{1-e^z} + 1\right) dz$$

$$= z - \log(1-e^z)$$

$$\therefore \quad w = \{z - \log(1-e^z)\} \bar{e}^z$$

Thus, P.I. $= \{z - \log(1-e^z)\} \bar{e}^z$

so, the general solution is

$$y = (c_1 + c_2 z)\bar{e}^z + \{z - \log(1-e^z)\} \bar{e}^z$$

$$= \frac{1}{x}\left[(c_1 + c_2 \log x) + \log \frac{x}{1-x}\right]$$

Another Process :-

$$\text{P.I.} = \frac{1}{D+1}\left[\frac{1}{D+1}\left(\frac{1}{(1-e^z)^2}\right)\right]$$

$$= \frac{1}{D+1}\left[\bar{e}^z \int \frac{e^z}{(1-e^z)^2} dz\right]$$

$$= \frac{1}{D+1} \frac{\bar{e}^z}{1-e^z} = \bar{e}^z \int \frac{e^z \cdot \bar{e}^z}{1-e^z} dz$$

$$= -\bar{e}^z \int \frac{dz}{e^z - 1}$$

$$= \bar{e}^z \cdot \int \left(1 + \frac{e^z}{1-e^z}\right) dz = \bar{e}^z \{z - \log(1-e^z)\}$$

$$= \bar{x}^1 [\log x - \log(1-x)] = \bar{x}^1 \log \frac{x}{1-x}$$

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Ref :- 1. Dr. Arup Mukherjee and Bej, Ordinary and Partial Differential Equations.

2. Ordinary and Partial differential equations, Maity & Ghosh.

Ex.3 $(x+a)^2 \frac{d^2y}{dx^2} - 4(x+a)\frac{dy}{dx} + 6y = x$.

$\Rightarrow$ Put $x+a = e^z \Rightarrow z = \log(x+a)$ and $\frac{dz}{dx} = \frac{1}{x+a}$

Now, $\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{1}{x+a}\cdot\frac{dy}{dz}$ ——①

and $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x+a}\cdot\frac{dy}{dz}\right) = \frac{-1}{(x+a)^2}\frac{dy}{dz} + \frac{1}{x+a}\cdot\frac{d}{dz}\left(\frac{dy}{dz}\right)\cdot\frac{dz}{dx}$

$= -\frac{1}{(x+a)^2}\cdot\frac{dy}{dz} + \frac{1}{(x+a)^2}\cdot\frac{d^2y}{dz^2} = \frac{1}{(x+a)^2}\left[\frac{d^2y}{dz^2} - \frac{dy}{dz}\right]$ ——②

Thus from ① & ② we get,

$(x+a)\frac{dy}{dx} = \frac{dy}{dz}$ and $(x+a)^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$

Now, the given equation becomes,

$\frac{d^2y}{dz^2} - \frac{dy}{dz} - 4\frac{dy}{dz} + 6y = e^z + a$

or, $\frac{d^2y}{dz^2} - 5\frac{dy}{dz} + 6y = e^z + a$

or, $(D^2 - 5D + 6)y = e^z + a$, where $\frac{d}{dz} \equiv D$.

∴ The A.E. is $(m-2)(m-3) = 0$ or, $m = 2, 3$,

∴ C.F. $= c_1e^{2z} + c_2e^{3z} = c_1(x+a)^2 + c_2(x+a)^3$, where c_1 and c_2 arbitrary constants.

Now, P.I. $= \frac{1}{D^2 - 5D + 6}(e^z + a)$

$= \frac{1}{D^2 - 5D + 6}e^z + \frac{1}{D^2 - 5D + 6}a$

$= \frac{1}{2}e^z - \frac{1}{6}\left(1 + \frac{D^2 - 5D}{6}\right)^{-1}a$

$= \frac{1}{2}e^z - \frac{1}{6}\left(1 - \frac{D^2 - 5D}{6} + \cdots\right)a$

$= \frac{1}{2}e^z - \frac{1}{6}a = \frac{1}{2}(x+a) - \frac{1}{6}a$

So, the general solution is

$y = c_1(x+a)^2 + c_2(x+a)^3 + \frac{1}{2}(x+a) - \frac{1}{6}a$.

Ex.4 :- $(1+2x)^2\frac{d^2y}{dx^2} - 6(1+2x)\frac{dy}{dx} + 16y = 8(1+2x)^2$

$\Rightarrow$ Put, $1+2x = e^z$

Therefore, $z = \log(1+2x)$ and $\frac{dz}{dx} = \frac{2}{1+2x}$

Now, $\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{2}{1+2x}\cdot\frac{dy}{dz}$ ——①

and $\frac{d^2y}{dx^2} = \frac{d}{dx}\left\{\frac{2}{1+2x}\cdot\frac{dy}{dz}\right\}$

$= \frac{2}{1+2x}\cdot\frac{d^2y}{dz^2}\cdot\frac{dz}{dx} - \frac{4}{(1+2x)^2}\cdot\frac{dy}{dz}$

$= \frac{4}{(1+2x)^2}\cdot\frac{d^2y}{dz^2} - \frac{4}{(1+2x)^2}\cdot\frac{dy}{dz}$

$= \frac{4}{(1+2x)^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$ ——— (2)

Now, from (1) and (2) we get,

$(1+2x)\frac{dy}{dx} = 2\frac{dy}{dz}$ and $(1+2x)^2\frac{d^2y}{dx^2} = 4\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$

Now, the given equation becomes,

$4\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) - 12\frac{dy}{dz} + 16y = 8e^{2z}$

or, $\frac{d^2y}{dz^2} - 4\frac{dy}{dz} + 4y = 2e^{2z}$

or, $(D^2 - 4D + 4)y = 2e^{2z}$, where $D \equiv \frac{d}{dz}$

or, $(D-2)^2 y = 2e^{2z}$

$\therefore$ A.E. $= (m-2)^2 = 0$ i.e, $m = 2, 2$,

Thus, C.F. $= (c_1 + c_2 z)e^{2z}$, where c_1, c_2 are arbitrary constants.

P.I. $= \frac{1}{(D-2)^2}2e^{2z} = e^{2z}\frac{1}{(D+2-2)^2}(2)$

$= e^{2z}\cdot\frac{1}{D^2}(2) = z^2e^{2z}$

Therefore, the complete solution is

$y = (c_1 + c_2 z)e^{2z} + z^2e^{2z}$

$= \{c_1 + c_2 \log(1+2x)\}(1+2x)^2 + (1+2x)^2\{\log(1+2x)\}^2$

$= (1+2x)^2\left[\{c_1 + c_2\log(1+2x)\} + \{\log(1+2x)\}^2\right]$

Exercise :

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1. $x^2y'' - 4xy' + 6y = \log x^2$
2. $x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x\log x$
3. $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x$
4. $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2x^2$
5. $(2+x)^2\frac{d^2y}{dx^2} + (2+x)\frac{dy}{dx} + 4y = 2\sin\{2\log(2+x)\}$

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P.P.