Prof. Surajit Dhara

SACT,

Dept. Of Physics, Narajole Raj College

DSC1BT (Electricity and Magnetism) , Topic :- Magnetism

Magnetostatics

The Divergence of Magnetic Field:

According to Biot- Savart Law the magnetic field B at any point $P\ (x, y, z)$ due to a volume distribution of current $J(x', y', z')$ is-

$$\vec{B} = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J} X \vec{R}\ dV}{R^3}$$

Where, $dV = dx'dy'dz'$

$$\vec{\nabla}.\vec{B} = \frac{\mu_0}{4\pi}\int_V \vec{\nabla}.(\frac{\vec{J} X \vec{R}}{R^3})dV$$

$\vec{\nabla}.\left(\frac{\vec{J} X \vec{R}}{R^3}\right) = \frac{\vec{R}}{R^3}.(\vec{\nabla} X \vec{J})$ - $\vec{J}.(\vec{\nabla} X \frac{\vec{R}}{R^3})$

$\vec{\nabla} X \frac{\vec{R}}{R^3} = \frac{1}{R^3}(\vec{\nabla} X \vec{R}) + \vec{\nabla}\frac{1}{R^3}\ X\ \vec{R}$

Here, $\vec{\nabla} X \vec{R} = 0$, and $\vec{\nabla}\frac{1}{R^3} = -\frac{3\vec{R}}{R^5}$

Thus, $\vec{J}.\left(\vec{\nabla} X \frac{\vec{R}}{R^3}\right) = 0$

Hence, $\vec{\nabla}.\vec{B} = 0$

This result Gauss's Law in Magnetostatics, $i.e.$ $\int_V \vec{\nabla}.\vec{B}\ dV = \oint_S \vec{B}.\vec{dS} = 0$

The Curl of Magnetic Field:

$\vec{B} = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J} X \vec{R}\ dV}{R^3}$............(1)

Where, $dV = dx'dy'dz'$ and $\vec{\nabla}\frac{1}{R} = -\frac{\vec{R}}{R^3}$

$$\vec{B} = -\frac{\mu_0}{4\pi}\int \vec{J}\ X\ \vec{\nabla}\frac{1}{R}\ dV\ldots\ldots\ldots\ldots(2)$$

$$\vec{\nabla}\ X\frac{1}{R}\vec{J} = \frac{1}{R}\vec{\nabla}\ X\vec{J} + \vec{\nabla}\frac{1}{R}\ X\vec{R} = -\vec{J}\ X\ \vec{\nabla}\frac{1}{R}$$

Therefore, $\vec{B} = \frac{\mu_0}{4\pi}\vec{\nabla}\ X\int\frac{1}{R}\vec{J}\ dV$

We know that $\vec{\nabla}\ X\left(\vec{\nabla}\ X\vec{A}\right) = \vec{\nabla}\left(\vec{\nabla}.\vec{A}\right) - \nabla^2\vec{A}$

$$\vec{\nabla}\ X\vec{B} = \frac{\mu_0}{4\pi}\ \vec{\nabla}\ (\vec{\nabla}.\int\overrightarrow{\frac{1}{R}J}\ dV) - \frac{\mu_0}{4\pi}\nabla^2\int\overrightarrow{\frac{1}{R}J}\ dV$$

$$= \frac{\mu_0}{4\pi}\ \vec{\nabla}\ \int\vec{J}.\vec{\nabla}\frac{1}{R}\ dV) - \frac{\mu_0}{4\pi}\int\vec{J}\ \nabla^2\frac{1}{R}dV$$

$\nabla^2\frac{1}{R} = -4\pi\delta(\vec{r} - \vec{r}')$ and $\vec{\nabla}\frac{1}{R} = -\vec{\nabla}'\frac{1}{R}$

Therefore, $\vec{\nabla}\ X\vec{B} = -\frac{\mu_0}{4\pi}\int\vec{J}.\vec{\nabla}'\frac{1}{R}\ dV + \frac{\mu_0}{4\pi}\int\vec{J}\ 4\pi\delta(\vec{r} - \vec{r}')\ dV$

$$= -\frac{\mu_0}{4\pi}\vec{\nabla}\int[\vec{\nabla}'.\frac{1}{R}\vec{J} - \frac{1}{R}\ \vec{\nabla}'.\vec{J}\ ]dV + \mu_0\vec{J}\ (\vec{r}\ )$$

For steady current

$$\vec{\nabla}\ X\vec{B} = \mu_0\vec{J}\ (\vec{r}\ ) - \frac{\mu_0}{4\pi}\vec{\nabla}\int[\vec{\nabla}'.\frac{1}{R}\vec{J}]dV$$

The integral on the right hand side can be converted using divergence theorem to a surface integral.

$$\vec{\nabla}\ X\vec{B} = \mu_0\vec{J}\ (\vec{r}\ )$$

$$\int_S\ (\vec{\nabla}\ X\vec{B}).\overrightarrow{dS} = \mu_0\int_S\ \vec{J}\ .\overrightarrow{dS}$$

Or, $\oint\overrightarrow{B}.\overrightarrow{dl} = \mu_0\ I$

Magnetic Vector Potential:

$$\vec{B} = \vec{\nabla} X\vec{A}$$

$\vec{\nabla} X\vec{B} = \mu_0 \vec{J}$

$\vec{\nabla} X (\vec{\nabla} X\vec{A}) = \mu_0 \vec{J}$

Or, $\vec{\nabla} (\vec{\nabla} .\vec{A}) - \nabla^2\vec{A} = \mu_0 \vec{J}$

We chose that $= \vec{\nabla} .\vec{A} = 0$ and $\nabla^2\vec{A} = -\mu_0 \vec{J}$ is called vector Poisson's equation.

$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho \, dV}{R}$.............(1)

$\vec{A} (\vec{r}') = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \, dV}{R}$.........(2)

The solution of (2) can be obtained from $\vec{B} = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J} X\vec{R} \, dV}{R^3}$.....(3)

$\vec{\nabla}\frac{1}{R} = -\frac{\vec{R}}{R^3}$ and $\vec{\nabla} X\frac{1}{R}\vec{J} - \frac{1}{R}\vec{\nabla} X\vec{J}$

$\vec{J} X \frac{\vec{R}}{R^3} = \vec{\nabla} X\frac{1}{R}\vec{J}$ ....(4)

Therefore, from equn. (3) and (4)

$\vec{B} = \vec{\nabla} X \frac{\mu_0}{4\pi} \int \frac{\vec{J}}{R} dV$

$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}}{R} dV$

Derivation of Biot-Savart Law and Ampere's Law from the Concept of Magnetic Vector Potential:

Biot-Savart Law:

The magnetic vector potential $\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}}{R} dV$

$$\vec{B} = \vec{\nabla} X\vec{A} = \frac{\mu_0}{4\pi} \int \vec{\nabla} X \frac{\vec{J}}{R} dV$$

$\vec{\nabla} X \frac{1}{R}\vec{J} = \frac{1}{R}\vec{\nabla} X\vec{J} + \vec{\nabla}\frac{1}{R} X\vec{R} = -\vec{J} X \vec{\nabla}\frac{1}{R} = -\frac{\vec{R}}{R^3} X \vec{J}$

Therefore, $\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{J} X \vec{R}}{R^3} dV$

And each volume current element $\vec{J}\,dV$ contributes the quantity-

$$d\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\;X\;\vec{R}}{R^3}dV$$

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\vec{dl}\;X\vec{R}}{R^3}$$

This is known as $Biot - Savart\ Law.$

Ampere's Law:

$$\oint \vec{B}.\vec{dl} = \oint_C (\vec{\nabla}\,X\vec{A}).\vec{dl} = \iint \quad \vec{\nabla}\,X(\vec{\nabla}\,X\vec{A}).\vec{dS}$$

We know that $\vec{\nabla}\,X\left(\vec{\nabla}\,X\vec{A}\right) = \vec{\nabla}\left(\vec{\nabla}.\vec{A}\right) - \nabla^2\vec{A} = \mu_0\vec{J}$

We choose, $\vec{\nabla}.\vec{A} = 0$

$\oint_C \vec{B}.\vec{dl} = \iint \mu_0\vec{J}\;.\vec{dS} = \mu_0 I$, which is Ampere's circuital Law.

Calculation of A for straight current carrying conductor:

We consider a straight wire AB carrying a steady current I. We want to calculate vector potential $\vec{A}$ at any point P. We consider an element $\vec{dl}$=$\boldsymbol{z}\,dz$

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{I\vec{d}}{R}$$

$$\vec{A} = \boldsymbol{z}\frac{\mu_0 I}{4\pi}\int_{-l_1}^{+l_2}\frac{dz}{\sqrt{r^2+z^2}}$$

$$\vec{A} = \boldsymbol{z}\frac{\mu_0 I}{4\pi}[\ln(z+\sqrt{r^2+z^2})]_{-l_1}^{+l_2}$$

$$\vec{A} = \boldsymbol{z}\frac{\mu_0 I}{4\pi}\ln[\frac{L+\sqrt{r^2+L^2}}{-L+\sqrt{r^2+L^2}}]$$

$$\vec{A} == \boldsymbol{z}\frac{\mu_0 I}{4\pi}\,ln\,[\frac{1+(1+\frac{r^2}{L^2})^{\frac{1}{2}}}{-1+(1+\frac{r^2}{L^2})^{\frac{1}{2}}}]$$

$$\vec{A} == \boldsymbol{z}\frac{\mu_0 I}{4\pi}\ln[\frac{2+\frac{r^2}{2L^2}}{\frac{r^2}{2L^2}}]$$

$$\vec{A} == \boldsymbol{z}\frac{\mu_0 I}{4\pi}\ln[\frac{4L^2}{r^2}+1]$$

$$\vec{A} == \boldsymbol{z}\frac{\mu_0 I}{4\pi}\ln(\frac{2L}{r})$$

We know that

$$\vec{B} = \vec{\nabla}\, \mathrm{X}\vec{A}$$

$$= \frac{1}{r}\begin{vmatrix} \boldsymbol{r} & r\boldsymbol{\theta} & \boldsymbol{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & 0 & A_z \end{vmatrix} = -\boldsymbol{\theta}\frac{\partial A_z}{\partial r} = -\boldsymbol{\theta}\frac{\partial}{\partial r}\,[\frac{\mu_0 I}{2\pi}\ln(\frac{2L}{r})]$$

$$= \boldsymbol{\theta}\frac{\mu_0 I}{2\pi r}$$

2. Two long straight parallel wires carrying same current:

Suppose we have two long parallel wires each of length 2L carrying current I in opposite directions. The magnetic vector potential $\vec{A} = \boldsymbol{A_1} + \boldsymbol{A_2}$

$$= \boldsymbol{z}\frac{\mu_0 I}{2\pi}\ln(\frac{2L}{r_1}) - \boldsymbol{z}\frac{\mu_0 I}{2\pi}\ln(\frac{2L}{r_2})$$

$$= \boldsymbol{z}\frac{\mu_0 I}{2\pi}\ln(\frac{r_2}{r_1})$$

Here, r_1 and r_2 are the distances of the point of observation P from the wires and $\boldsymbol{z}$ is a unit vector parallel to the wires.

Torque acting in small current loop:

Force on a current element $I\vec{dl} = \overrightarrow{dF} = I\vec{dl}\, X\, \vec{B}$

$$\overrightarrow{d\tau} = \vec{r}\, X\, \overrightarrow{dF} = \overrightarrow{dF} = I\vec{r}\, X\, (\vec{dl}\, X\, \vec{B})$$

$$From\ vector\ analysis -$$

$$\vec{r}\, X\, (\vec{dl}\, X\, \vec{B}) + \vec{dl}\, X\, (\vec{B}\, X\, \vec{r}) + \vec{B}\, X\, (\overrightarrow{r\, X\, dl}) = 0$$

$$\vec{r}\, X\, (\vec{dl}\, X\, \vec{B}) = -\vec{dl}\, X\, (\vec{B}\, X\, \vec{r}) - \vec{B}\, X\, (\overrightarrow{r\, X\, dl})$$

We know that-

$$\overrightarrow{d[r\, X\, (\vec{r}\, X\, \vec{B})]} = \overrightarrow{dr}\, X\, (\vec{r}\, X\, \vec{B}) + \vec{r}\, X\, (d\vec{r}\, X\, \vec{B})$$

Since, $d\vec{r} = d\vec{l}$ and $\vec{B} = uniform$

$$\vec{r}\, X\, (\vec{dl}\, X\, \vec{B}) = d[\vec{r}\, X\, (\vec{r}\, X\, \vec{B}) - \vec{dl}\, X\, (\overrightarrow{r\, X\, B})$$

$$\overrightarrow{2r}\, X\, (\vec{dl}\, X\, \vec{B}) = d[\vec{r}\, X\, (\vec{r}\, X\, \vec{B})] - \vec{B}\, X\, (\overrightarrow{r\, X\, dl})$$

Therefore, $\overrightarrow{d\tau} = \frac{1}{2} I\{d[\vec{r}\, X\, (\vec{r}\, X\, \vec{B})] - \vec{B}\, X\, (\overrightarrow{r\, X\, dl})\}$

Total torque on the loop is $\overrightarrow{d\tau} = \frac{1}{2} I\{\oint\{d[\vec{r}\, X\, (\vec{r}\, X\, \vec{B})] - \vec{B}\, X \oint(\overrightarrow{r\, X\, dr})\}$

$$\vec{\tau} = \frac{1}{2} I \oint (\overrightarrow{r\, X\, dr})\, X\, \vec{B}$$

$$\vec{\tau} = \vec{m}\, X \vec{B}$$

$$\vec{m} = \frac{1}{2} I \oint (\overrightarrow{r\, X\, dr})$$