Dr. Avradip Pradhan,
Assistant Professor,
Department of Physics,
Narajole Raj College, Narajole.

DSC1BT (Electricity and Magnetism)

Topic – Maxwell's Equation and EM Wave Propagation (Part – 2)

We have already discussed part 1 of this e-report.

Now let us continue part 2 of it.

EM Wave Propagation in Vacuum:

In regions of free space or vacuum where there is no charge ($\rho = 0$) or current ($\vec{J} = 0$), all the four Maxwell's equations can be written as

$$\nabla.\vec{E} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla.\vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

They constitute a set of coupled, first-order partial differential equations for $\vec{E}$ and $\vec{B}$. They can be decoupled by applying the curl to the 2nd and 4th equations.

Therefore, we get

$$\nabla \times (\nabla \times \vec{E}) = -\nabla \times \frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times \vec{B})$$

Using elementary vector algebra, we write $\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla.\vec{E}) - \nabla^2 \vec{E}$, which will be reduced to $-\nabla^2 \vec{E}$ (since $\nabla.\vec{E} = 0$).

Dr. Avradip Pradhan,
Assistant Professor,
Department of Physics,
Narajole Raj College, Narajole.

So, we get $-\nabla^2\vec{E} = -\mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$ or

$$\nabla^2\vec{E} = \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

In a similar process and using the fact that $\nabla.\vec{B} = 0$, we obtain

$$\nabla^2\vec{B} = \mu_0\varepsilon_0\frac{\partial^2\vec{B}}{\partial t^2}$$

We now have separate equations for $\vec{E}$ and $\vec{B}$, but they are of second order, that's the price we paid for decoupling them. In vacuum then, each Cartesian component of $\vec{E}$ and $\vec{B}$ satisfies the three dimensional wave equation, given by

$$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

where v is the speed of the corresponding wave. So Maxwell's Equations imply that empty space supports the propagation of electromagnetic waves, travelling at a speed given by

$$v = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 3\times 10^8 \text{ ms}^{-1}$$

which happens to be precisely the velocity of light, denoted as c. The implication is astounding, perhaps light is an electromagnetic wave.

EM Wave Propagation in a Linear Dielectric Medium:

Inside a dielectric medium, but in regions where there is no free charge or free current, Maxwell's equations become

$$\nabla.\vec{D} = 0$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\nabla.\vec{B} = 0$$

Dr. Avradip Pradhan,
Assistant Professor,
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$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$$

For a linear medium $\vec{D} = \varepsilon \vec{E}$, and $\vec{H} = \frac{1}{\mu}\vec{B}$, where ε and μ are the electric permittivity and magnetic permeability of the medium respectively. Moreover, our assumption will be for a homogeneous medium as well. Therefore, ε and μ will be independent both of space and time. In that case, Maxwell's Equations reduce to

$$\nabla . \vec{E} = 0 \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla . \vec{B} = 0 \qquad \nabla \times \vec{B} = \mu\varepsilon \frac{\partial \vec{E}}{\partial t}$$

Therefore, these equations will finally reduce to

$$\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} \qquad \nabla^2 \vec{B} = \mu\varepsilon \frac{\partial^2 \vec{B}}{\partial t^2}$$

Therefore inside a medium, light or EM wave travels with a speed

$$v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{n} \text{ with } n = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}}$$

where n is called as the *refractive index* of the medium. Obviously, for vacuum $n = 1$.

For most of the material $\mu \approx \mu_0$. Therefore $n \approx \sqrt{\frac{\varepsilon}{\varepsilon_0}} = \sqrt{\varepsilon_r}$, with ε_r as the *relative permittivity* or *dielectric constant* of the medium.

Transverse Nature of EM Waves:

We now confine our attention to sinusoidal waves of frequency ω. Since different frequencies in the visible range correspond to different colours, such waves are called *monochromatic*. Let us suppose that the waves are travelling in the z direction and have no x or y dependence, these are called *plane waves*,

Dr. Avradip Pradhan,
Assistant Professor,
Department of Physics,
Narajole Raj College, Narajole.

because the fields are uniform over every plane perpendicular to the direction of propagation (shown in Fig. 1). We are interested then, in fields of the form

$$\vec{E}(z,t) = \vec{E}_0 e^{i(kz-\omega t)}$$

$$\vec{B}(z,t) = \vec{B}_0 e^{i(kz-\omega t)}$$

where $\vec{E}_0$ and $\vec{B}_0$ are the complex amplitudes and $\omega = ck$.

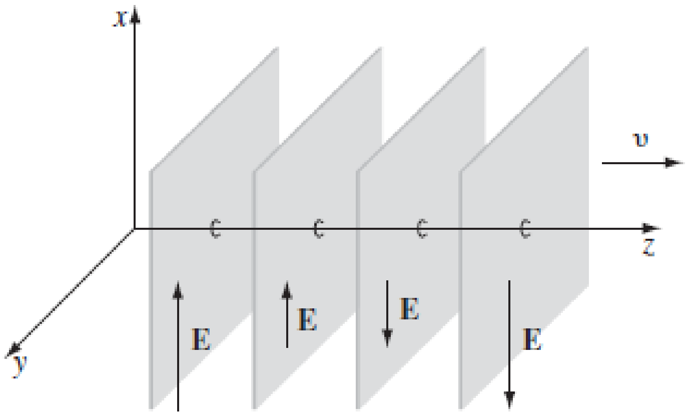

Fig. 1

Now, the wave equations for $\vec{E}$ and $\vec{B}$ were derived from Maxwell's Equations. However, whereas every solution to Maxwell's Equations in free space must obey the wave equation, the converse is not true, which means Maxwell's Equations impose extra constraints on $\vec{E}_0$ and $\vec{B}_0$. In particular, since $\nabla.\vec{E} = 0$ and $\nabla.\vec{B} = 0$, it follows that

$$ik\left(\vec{E}_0\right)_z e^{i(kz-\omega t)} = 0$$

$$ik\left(\vec{B}_0\right)_z e^{i(kz-\omega t)} = 0$$

which give us $\left(\vec{E}_0\right)_z = \left(\vec{B}_0\right)_z = 0$. So, the electric and magnetic fields do not have any components along the direction of propagation, rather they are perpendicular to the direction of propagation. That is, electromagnetic waves are *transverse* in nature.

Dr. Avradip Pradhan,
Assistant Professor,
Department of Physics,
Narajole Raj College, Narajole.

Moreover, Faraday's Law $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ implies a relation between the electric and magnetic amplitudes

$$-ik\left(\vec{E}_0\right)_y e^{i(kz-\omega t)} = i\omega\left(\vec{B}_0\right)_x e^{i(kz-\omega t)} \text{ and}$$

$$ik\left(\vec{E}_0\right)_x e^{i(kz-\omega t)} = i\omega\left(\vec{B}_0\right)_y e^{i(kz-\omega t)}.$$

So, we write $\left(\vec{B}_0\right)_x = -\frac{k}{\omega}\left(\vec{E}_0\right)_y$ and $\left(\vec{B}_0\right)_y = \frac{k}{\omega}\left(\vec{E}_0\right)_x$, which can be generalized as

$$\vec{B}_0 = \frac{k}{\omega}(\hat{z} \times \vec{E}_0)$$

Evidently, $\vec{E}$ and $\vec{B}$ are *in phase* and *mutually perpendicular*, their real amplitudes are related by

$$\frac{B_0}{E_0} = \frac{k}{\omega} = \frac{1}{c}.$$

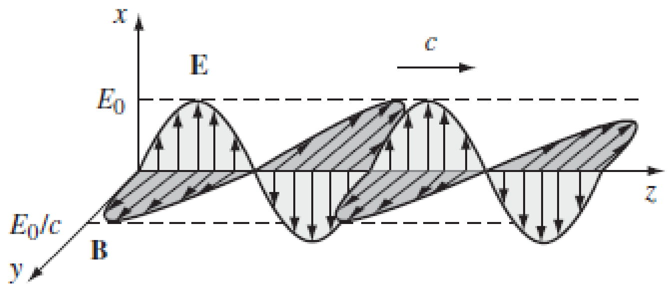

Fig. 2

If the electric field $\vec{E}$ points in the x direction, then the magnetic field $\vec{B}$ points in the y direction, we can write

$$\vec{E}(z,t) = E_0 e^{i(kz-\omega t)}\hat{x} \text{ and } \vec{B}(z,t) = \frac{1}{c}E_0 e^{i(kz-\omega t)}\hat{y}$$

This is the paradigm for a monochromatic plane wave (Fig. 2). The wave as a whole is said to be polarized in the x direction (by convention, we use the direction of $\vec{E}$ to specify the polarization of an electromagnetic wave). There is

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Assistant Professor,
Department of Physics,
Narajole Raj College, Narajole.

nothing special about the z direction, of course—we can easily generalize to monochromatic plane waves travelling in an arbitrary direction. The notation is facilitated by the introduction of the *propagation vector* (or *wave vector*) $\vec{k}$, pointing in the direction of propagation, whose magnitude is the wave number k. So, in general, it can be written as

$$\vec{E}(\vec{r},t) = E_0 e^{i(\vec{k}.\vec{r}-\omega t)}\hat{n}$$

$$\vec{B}(\vec{r},t) = \frac{1}{c}E_0 e^{i(\vec{k}.\vec{r}-\omega t)}(\hat{k}\times\hat{n}) = \frac{1}{c}(\hat{k}\times\vec{E})$$

where $\hat{n}$ is the polarization vector. Because $\vec{E}$ is transverse in nature, we have

$$\hat{n}.\hat{k} = 0$$

Polarization:

Polarization is a property applying to transverse waves that specifies the geometrical orientation of the oscillations of the wave. An electromagnetic or EM wave such as light consists of a coupled oscillating electric field and magnetic field which are always perpendicular to each other. By convention, the polarization of electromagnetic waves refers to the direction of the electric field. It is an important parameter in areas of science dealing with transverse waves, such as optics, seismology, radio, and microwaves. Especially impacted are technologies such as lasers, wireless and optical fibre telecommunications and radar. The simplest electromagnetic field type is that of a plane wave. Then each Cartesian component of the field (electric as well as magnetic) vectors and consequently the vectors $\vec{E}$ and $\vec{H}$ themselves are functions of the variable $\tau = \omega t - \vec{k}.\vec{r}$ only.

Different types of Polarization. Of particular interest is the case when the plane wave is time-harmonic, i.e. when each Cartesian component of $\vec{E}$ and $\vec{H}$ is of the form

$$a\cos(\tau+\delta) = Re\{ae^{-i(\tau+\delta)}\} \text{ with } a > 0$$

Here τ denotes the variable part of the phase factor, i.e. $\tau = \omega t - \vec{k}.\vec{r}$

Dr. Avradip Pradhan,
Assistant Professor,
Department of Physics,
Narajole Raj College, Narajole.

Now we discuss different types of polarization. We choose the z −axis in the propagation direction. Then, since the field is transverse, only the x and y components of $\vec{E}$ and $\vec{H}$ will be non-zero. We shall now consider the nature of the curve which the end point of the electric vector describes at a typical point in space. This curve is the locus of the points whose coordinates (E_x, E_y) are given by

$$E_x = a_1 \cos(\tau + \delta_1)$$

$$E_y = a_2 \cos(\tau + \delta_2)$$

Elliptic Polarization. In order to eliminate τ between the first two equations of, we re-write them in the form

$$\frac{E_x}{a_1} = \cos(\tau + \delta_1) = \cos\tau \cos\delta_1 - \sin\tau \sin\delta_1$$

$$\frac{E_y}{a_2} = \cos(\tau + \delta_2) = \cos\tau \cos\delta_2 - \sin\tau \sin\delta_2$$

After a little bit of algebra we obtain

$$\left(\frac{E_x}{a_1}\right)^2 + \left(\frac{E_y}{a_2}\right)^2 - 2\left(\frac{E_x}{a_1}\right)\left(\frac{E_y}{a_2}\right)\cos\delta = \sin^2\delta$$

where $\delta = \delta_2 - \delta_1$. This gives the equation of an ellipse. The ellipse is inscribed in a rectangle whose sides are parallel to the coordinate axes and whose lengths are $2a_1$ and $2a_2$ (as shown in Fig. 3). The ellipse touches the sides at the point $(\pm a_1, \pm a_2 \cos\delta)$ and $(\pm a_1 \cos\delta, \pm a_2)$.

The wave in this case is said to be *elliptically polarized*. In general the axes of the ellipse are not in the x and y directions. Let ξ and η be a new set of axes along the axes of the ellipse and let ψ be the angle between x and the direction ξ of the major axis (Fig. 3). Then the components E_ξ and E_η are related to E_x and E_y by

$$E_\xi = E_x \cos\psi + E_y \sin\psi$$

$$E_\eta = -E_x \sin\psi + E_y \cos\psi$$

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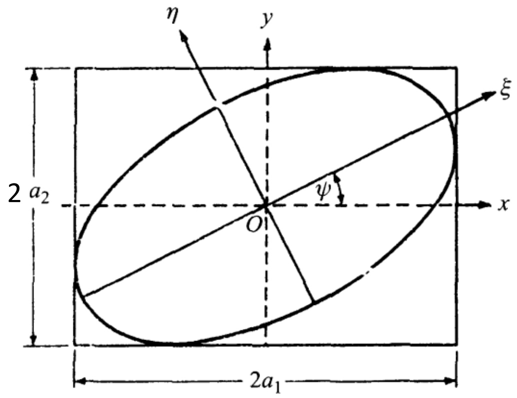

Fig. 3

We distinguish two cases of polarization, according to the sense in which the end point of the electric vector describes the ellipse. It seems natural to call the polarization right-handed or left-handed according to whether the rotation of $\vec{E}$ and the direction of propagation form a right-handed or left-handed screw. But the traditional terminology is just the opposite — being based on the apparent behaviour of $\vec{E}$ when 'viewed' face on by the observer. Thus we say that the polarization is *right-handed* when to an observer looking in the direction from which the light is coming, the end point of the electric vector would appear to describe the ellipse in the *clockwise* sense. We see that in this case $\sin\delta > 0$. For *left-handed* polarization the opposite is the case, i.e. to an observer looking in the direction from which the light is propagated, the electric vector would appear to describe the ellipse *anticlockwise* and in this case $\sin\delta < 0$.

Linear Polarization and Circular Polarization. Two special cases are of particular importance, namely when the polarization ellipse degenerates into a straight line or a circle.

The ellipse will reduce to a straight line when $\delta = \delta_2 - \delta_1 = m\pi$ where m is any positive or negative integer. There we obtain $\frac{E_y}{E_x} = (-1)^m \frac{a_2}{a_1}$, which is an equation of a straight line. We say that the wave is *linearly polarized* (or less frequently *plane polarized*).

The other special case of importance is that of a *circularly polarized* wave, the ellipse then degenerating into a circle. Clearly a necessary condition for this is

Dr. Avradip Pradhan,
Assistant Professor,
Department of Physics,
Narajole Raj College, Narajole.

that the circumscribed rectangle shall become a square with sides $a_1 = a_2 = a$. And also it is required that $\delta = \delta_2 - \delta_1 = \frac{m\pi}{2}$ with m any positive or negative odd integer. Here we obtain the equation of a circle given by ${E_x}^2 + {E_y}^2 = a^2$.

If instead of the real representation, the complex one is used (i.e. if the exponential instead of the cosine function is written), then

$$\frac{E_y}{E_x} = \frac{a_2}{a_1} e^{-i\delta}$$

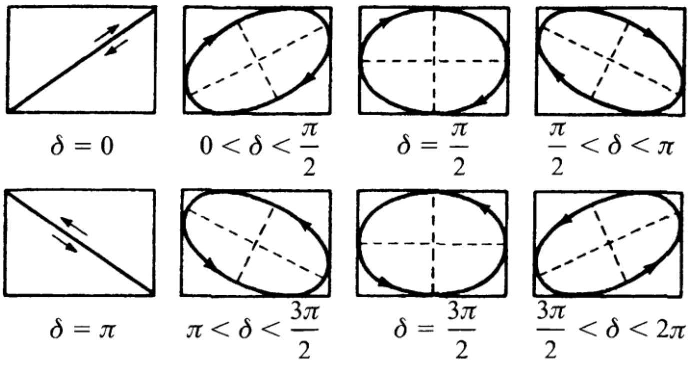

Fig. 4

From the value of this ratio one can immediately determine the nature of the polarization. For example, we obtain

(i) Linearly polarized electric wave ($\delta = m\pi, m = 0, \pm 1, \pm 2, \ldots$)

$$\frac{E_y}{E_x} = (-1)^m \frac{a_2}{a_1}$$

(ii) Right-handed circularly polarized electric wave ($a_1 = a_2, \delta = \frac{m\pi}{2}$)

$$\frac{E_y}{E_x} = e^{-\frac{im\pi}{2}} = -i$$

(iii) Left-handed circularly polarized electric wave ($a_1 = a_2, \delta = -\frac{m\pi}{2}$)

$$\frac{E_y}{E_x} = e^{\frac{im\pi}{2}} = i$$

Fig. 4 illustrates how the polarization ellipse changes with varying δ.

Dr. Avradip Pradhan,
Assistant Professor,
Department of Physics,
Narajole Raj College, Narajole.

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(All the figures have been collected from the above mentioned references)

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