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DSC1BT (Electricity and Magnetism)

Topic – Maxwell's Equation and EM Wave Propagation (Part – 1)

Introduction:

While studying electricity and magnetism, one soon becomes aware that a number of relationships are described by vector cross-products or right-hand rules. In other words, an occurrence of one sort produces a related, perpendicularly directed response. Of immediate interest is the fact that a time-varying electric ($\vec{E}$) field generates a magnetic ($\vec{B}$) field, which is everywhere perpendicular to the direction in which the electric field changes. In the same way, a time-varying $\vec{B}$ field generates an $\vec{E}$ field, which is everywhere perpendicular to the direction in which the magnetic field changes. Consequently, we might anticipate the general transverse nature of the electric as well as magnetic fields in an electromagnetic disturbance.

Let us now consider a charge that is somehow caused to accelerate from rest. When the charge is at rest, it has associated with it a constant radial electric field extending in all directions presumably to infinity. At the instant the charge begins to move, the electric field is altered in the vicinity of the charge, and this alteration propagates out into space at some finite speed. The time-varying electric field induces a magnetic field by means of Maxwell's Equations. If the charge's velocity is constant, the rate-of-change of the electric field is steady, and the resulting magnetic field is constant. But here the charge is accelerating. Therefore $\frac{\partial \vec{E}}{\partial t}$ is itself not constant, so the induced magnetic field is time-dependent. The time-varying magnetic field generates an electric field, obtained from Maxwell's Equations and the process continues, with electric and magnetic coupled in the form of a pulse. As one field changes, it generates a new field that extends a bit farther, and the pulse moves out from one point to the next through space.

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We can draw an overly mechanistic but rather picturesque analogy, if we imagine the electric field lines as a dense radial distribution of strings. When somehow plucked, each string is distorted, forming a kink that travels outward from the source. All these kinks combine at any instant to yield a three-dimensional expanding pulse in the continuum of the electric field. The electric and magnetic fields can more appropriately be considered as two aspects of a single physical phenomenon, the *electromagnetic field* (or *EM field*, in short), whose source is a moving charge. The disturbance, once it has been generated in the electromagnetic field, is an untied wave that moves beyond its source and independently of it. Bound together as a single entity, the time-varying electric and magnetic fields regenerate each other in an endless cycle.

Equation of Continuity:

Let us begin by reviewing the *conservation of charge*, because it is the paradigm for all conservation laws. Here we need to distinguish between the *global conservation* of charge and a *local conservation* of charge. The global conservation tells us that the total charge in the universe is constant. But the local conservation is a much stronger statement, which says that if the charge in some region changes with time, then exactly that amount of charge must have passed in or out through the surface initially confining the charge. Let's now discuss the same point mathematically.

Formally, the charge (Q) in a given volume V at time t is expressed as

$$Q(t) = \int \rho(\vec{r}, t) dV$$

where $\rho(\vec{r}, t)$ is the volume charge density at a position $\vec{r}$ inside the given volume and at time t. If $\vec{J}$ is the current density then the current flowing out through the boundary of the volume is given by

$$I_{out} = \oint \vec{J}.d\vec{S}.$$

So, local conservation of charge says $\frac{dQ}{dt} = -I_{out}$

$$\text{or } \frac{d}{dt}\int \rho dV = -\oint \vec{J}.d\vec{S}$$

$$\text{or} \int \frac{\partial \rho}{\partial t} dV = -\oint \vec{J}.d\vec{S}$$

Using Divergence Theorem $\oint \vec{J}.d\vec{S} = \int (\nabla.\vec{J})\, dV$, we get

$$\int \frac{\partial \rho}{\partial t} dV = -\int (\nabla.\vec{J})\, dV$$

Since this expression is true for any arbitrary volume, it follows that

$$\frac{\partial \rho}{\partial t} + \nabla.\vec{J} = 0.$$

This is known as the *Equation of Continuity*, the precise mathematical statement of local conservation of charge. Conservation of charge is not an *independent* assumption; it is built into the laws of electrodynamics. It serves as a constraint on the sources (ρ and $\vec{J}$). They can't be just any arbitrary functions as they have to respect conservation of charge.

Displacement Current:

Before the discovery of Maxwell's Equations, we could encounter the following laws, specifying the divergence and curl of electric ($\vec{E}$) and magnetic ($\vec{B}$) fields. Those laws are given below

$$\nabla.\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla.\vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

Three of these four laws have specific names. The first law is known as Gauss's Law (or Gauss's Theorem), the second one is called as Faraday's Law, the third one doesn't have any name as such and the fourth one is known as Ampère's Circuital Law.

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Now, it happens that there is a fatal inconsistency in these formulae. It has to do with the old rule that divergence of curl is always zero. If we apply the divergence to the 2nd equation, everything works out, since we get

$$\nabla.\left(\nabla\times\vec{E}\right)=-\nabla.\frac{\partial\vec{B}}{\partial t}=-\frac{\partial}{\partial t}(\nabla.\vec{B})$$

The left side is zero because divergence of curl is always zero and the right side is zero by virtue of 3rd equation ($\nabla.\vec{B}=0$). But we you do the same thing to 4th equation, we find ourselves in trouble, since

$$\nabla.\left(\nabla\times\vec{B}\right)=\mu_0\nabla.\vec{J}$$

where the left side must be zero, but the right side is not, in general. For *steady currents*, the divergence of $\vec{J}$ is zero, but when we go beyond magnetostatics Ampère's law cannot be right.

Now applying the continuity equation and Gauss's law, the problematic term can be rewritten as

$$\mu_0\nabla.\vec{J}=-\mu_0\frac{\partial\rho}{\partial t}=-\mu_0\frac{\partial}{\partial t}\left(\varepsilon_0\nabla.\vec{E}\right)=-\mu_0\nabla.\left(\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$

$$\text{or }\mu_0\nabla.\left(\vec{J}+\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)=0$$

If we were to add up $\varepsilon_0\frac{\partial\vec{E}}{\partial t}$ with $\vec{J}$, in Ampère's law, it would be just right to erase the extra non-zero divergence. Maxwell called this extra term as the *displacement current* given by $\vec{J}_d=\varepsilon_0\frac{\partial\vec{E}}{\partial t}$. With this modification, the fourth law becomes

$$\nabla\times\vec{B}=\mu_0\left(\vec{J}+\vec{J}_d\right)=\mu_0\vec{J}+\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}.$$

Such a modification changes nothing, as far as magnetostatics is concerned. Since when $\vec{E}$ is constant in time, $\frac{\partial\vec{E}}{\partial t}=0$ and we still have $\nabla\times\vec{B}=\mu_0\vec{J}$, obtained before.

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Maxwell's Equations:

Incorporating all these corrections, Maxwell finally wrote down his classical electromagnetic equations, given by

$$\nabla.\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla.\vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Maxwell's Equations in Free Space. There is a pleasing symmetry to Maxwell's Equations. It is particularly striking in free space, where both ρ and $\vec{J}$ vanish. Therefore, using $\rho = 0$ and $\vec{J} = 0$, we get

$$\nabla.\vec{E} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla.\vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

If we replace $\vec{E}$ by $\vec{B}$ and $\vec{B}$ by $-\mu_0\varepsilon_0\vec{E}$, the first pair of equations turns into the second, and vice versa. Here lies the symmetry of Maxwell's Equations in between $\vec{E}$ and $\vec{B}$.

Maxwell's Equations in Matter. When we are working with materials that are subject to electric and magnetic polarization there is a more convenient way to write the Maxwell's Equations. For inside polarized matter there will be accumulations of bound charge and current (apart from free charge and current),

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over which we have no direct control. It would be nice to reformulate Maxwell's Equations so as to make explicit reference only to the free charges (ρ_f) and currents ($\vec{J}_f$).

We have already known, from the static case, that an electric polarization $\vec{P}$ produces a bound charge density given by

$$\rho_b = -\nabla.\vec{P}$$

Similarly a magnetization $\vec{M}$ results in a bound current given by

$$\vec{J}_b = \nabla \times \vec{M}$$

There's just one new feature to consider in the non-static case. Any change in the electric polarization involves a flow of bound charge (let's call it $\vec{J}_p$, the polarization current), which must be included in the total current. So, we have $\vec{J}_p = \frac{\partial \vec{P}}{\partial t}$. In view of all this, the total charge density can be separated into two parts as given by

$$\rho = \rho_f + \rho_b = \rho_f - \nabla.\vec{P}$$

and the current density into three parts

$$\vec{J} = \vec{J}_f + \vec{J}_b + \vec{J}_p = \vec{J}_f + \nabla \times \vec{M} + \frac{\partial \vec{P}}{\partial t}$$

With this, Gauss's Law can now be written as

$$\nabla.\vec{E} = \frac{\rho}{\varepsilon_0} = \frac{\rho_f - \nabla.\vec{P}}{\varepsilon_0}$$

$$\text{or } \nabla.\left(\varepsilon_0\vec{E} + \vec{P}\right) = \rho_f$$

$$\text{or } \nabla.\vec{D} = \rho_f$$

where $\vec{D} = \varepsilon_0\vec{E} + \vec{P}$ is the displacement vector for the medium.

Meanwhile, the modified Ampère's Law becomes

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$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \left(\vec{J}_f + \nabla \times \vec{M} + \frac{\partial \vec{P}}{\partial t} \right) + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\text{or } \nabla \times \left(\frac{\vec{B}}{\mu_0} - \vec{M} \right) = \vec{J}_f + \frac{\partial}{\partial t} \left(\varepsilon_0 \vec{E} + \vec{P} \right)$$

$$\text{or } \nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

where $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$ is the auxiliary magnetic field. Faraday's Law and $\nabla . \vec{B} = 0$ are not affected by our separation of charge and current into free and bound parts, since they do not involve ρ or $\vec{J}$.

In terms of free charges and currents, Maxwell's Equations are expressed as

$$\nabla . \vec{D} = \rho_f$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla . \vec{B} = 0$$

$$\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

Energy Density in Electromagnetic Field, Poynting Vector:

Let us suppose that we have some charge and current configuration which, at time t, produces fields $\vec{E}$ and $\vec{B}$. In the next instant dt, the charges move around a bit by $d\vec{l}$. According to the Lorentz force law, the work done on a charge q is

$$dW = \vec{F} . d\vec{l} = q\left(\vec{E} + \vec{v} \times \vec{B} \right) . \vec{v} dt = q\vec{E} . \vec{v} dt$$

So, the rate at which work is done on all the charges in a volume V is

$$\frac{dW}{dt} = q\vec{E} . \vec{v} = \int \rho \vec{E} . \vec{v} dV = \int \vec{E} . \vec{J} dV$$

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Since $\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$, we obtain $\vec{J} = \frac{1}{\mu_0} \nabla \times \vec{B} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$. Therefore, we can write

$$\vec{E}.\vec{J} = \frac{1}{\mu_0} \vec{E}.\left(\nabla \times \vec{B}\right) - \varepsilon_0 \vec{E}.\frac{\partial \vec{E}}{\partial t} = \frac{1}{\mu_0}\left\{\vec{B}.\left(\nabla \times \vec{E}\right) - \nabla.\left(\vec{E} \times \vec{B}\right)\right\} - \varepsilon_0 \vec{E}.\frac{\partial \vec{E}}{\partial t}$$

Using Faraday's Law $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, we write

$$\vec{E}.\vec{J} = \frac{1}{\mu_0}\left\{-\vec{B}.\frac{\partial \vec{B}}{\partial t} - \nabla.\left(\vec{E} \times \vec{B}\right)\right\} - \varepsilon_0 \vec{E}.\frac{\partial \vec{E}}{\partial t}$$

$$\text{or } \vec{E}.\vec{J} = -\frac{1}{\mu_0}\vec{B}.\frac{\partial \vec{B}}{\partial t} - \varepsilon_0 \vec{E}.\frac{\partial \vec{E}}{\partial t} - \frac{1}{\mu_0}\nabla.\left(\vec{E} \times \vec{B}\right)$$

$$\text{or } \vec{E}.\vec{J} = -\frac{1}{2}\frac{\partial}{\partial t}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \frac{1}{\mu_0}\nabla.\left(\vec{E} \times \vec{B}\right).$$

Therefore, the rate of work done is given as

$$\frac{dW}{dt} = -\frac{1}{2}\int \frac{\partial}{\partial t}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) dV - \frac{1}{\mu_0}\int \nabla.\left(\vec{E} \times \vec{B}\right) dV$$

Using divergence theorem on the 2nd integral, we obtain

$$\int \nabla.\left(\vec{E} \times \vec{B}\right) dV = \oint \left(\vec{E} \times \vec{B}\right).d\vec{S}$$

Therefore, $\frac{dW}{dt} = -\frac{d}{dt}\int \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) dV - \oint \frac{1}{\mu_0}\left(\vec{E} \times \vec{B}\right).d\vec{S}$.

Poynting Theorem. The first integral on the right is the total energy stored in the fields $\int u dV$ with $u = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)$ is the electromagnetic energy density. The last formula is basically the mathematical representation of so-called *Poynting's Theorem*, which says that the work done on the charges by the electromagnetic force is equal to the decrease in energy remaining in the fields, less the energy that flowed out through the surface.

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Here $\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B})$ is called the *Poynting Vector*, which is basically the energy per unit time, per unit area, transported by the fields or the energy flux density.

So, from Poynting Theorem, we proceed as

$$\frac{dW}{dt} = -\frac{d}{dt}\int u dV - \oint \vec{S}.d\vec{S}$$

In empty space, where there is no charge ($q = 0$), the work done will be zero. Therefore, using $\frac{dW}{dt} = 0$, we get

$$\frac{d}{dt}\int u dV = -\oint \vec{S}.d\vec{S}$$

$$\text{or } \int \frac{\partial u}{\partial t} dV = -\int (\nabla.\vec{S}) dV$$

and hence, $\frac{\partial u}{\partial t} + \nabla.\vec{S} = 0$. This is the basically the continuity equation for energy, where the energy density u plays the role of charge density ρ, and the Poynting vector $\vec{S}$ takes the part of current density $\vec{J}$. It expresses local conservation of electromagnetic energy.

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This concludes part 1 of this e-report.

The discussion will be continuing in the part 2 of this e-report.

Reference(s):

Introduction to Electrodynamics, D.J. Griffiths, Pearson

Optics, Eugene Hecht, Pearson Education

(All the figures have been collected from the above mentioned references)