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# C2T (Mechanics)

# Topic – Gravitation and Central Force Motion (Part – 3)

We have already discussed part 2 of this e-report.

Now let us continue part 3 of it.

## Kepler's Laws:

Johannes Kepler had the mathematical genius and talent to discover that the famous astronomer Tycho Brahe's accurate measurements about the planetary positions could be fitted by three simple empirical laws. The task was formidable. It took Kepler almost 18 years of laborious calculation to obtain the following three laws of planetary motion, which he stated early in the seventeenth century.

1. Every planet moves in an elliptic orbit with the sun at one of the foci.

2. The radius vector from the sun to a planet sweeps out equal areas in equal times, or the areal velocity for each planet is constant.

3. The square of the period of revolution $T$ of a planet about the sun is directly proportional to the cube of the major axis $A$ of the ellipse. Mathematically, it means $T^2 \propto A^3$ or $T^2 = kA^3$, where the proportionality constant $k$ is the same for all the planets.

Kepler's empirical laws went unexplained until the latter half of the seventeenth century, when Sir Isaac Newton's fascination with the problem of planetary motion inspired him to formulate his laws of motion and the law of universal gravitation. Using these mathematical laws, Newton explained Kepler's empirical laws, giving an overwhelming argument in favour of the new mechanics. Planetary motion and the more general problem of motion under a central force continue to play an important role in many branches of Physics and turn up in such topics as particle scattering, atomic structure and space navigation etc.

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## **Planetary Motion:**

Now we are going to solve the main problem of central force motion, which is finding the orbit for a planet of mass $m$ moving about a star of mass $M$ under the gravitational interaction

$$U(r) = -G\frac{Mm}{r} = -\frac{C}{r}$$

with $C = GMm$. Our results would also be applicable to a satellite of mass $m$ orbiting a planet of mass $M$ or even to a binary star system with two stars of masses $M$ and $m$. Further, we shall use our results to show how Newtonian mechanics can account for Kepler's empirical laws of planetary motion. Because we are often interested in a satellite of mass $m$ orbiting the earth (with mass $M_e$), it is very useful here to express $C$ in more familiar terms. At the earth's surface ($r = R_e$) the acceleration due to gravity is $g = G\frac{M_e}{{R_e}^2}$.

So, $C$ can be written as $C = GM_e m = mg{R_e}^2$ for a satellite orbiting the earth. Now, from the energy equation we obtain, $E = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} + U(r)$ and $L = \mu r^2\dot{\theta}$. From these two equations, we write

$$\frac{dr}{dt} = \sqrt{\frac{2}{\mu}[E - \frac{L^2}{2\mu r^2} - U(r)]}$$

$$\text{and } \frac{d\theta}{dt} = \frac{L}{\mu r^2}.$$

Often we are interested in the path of the particle, which means knowing $r$ as a function of $\theta$. We usually call the trajectory $r(\theta)$ as the *orbit* of the particle. Using the chain rule of derivative

$$\frac{d\theta}{dr} = \frac{d\theta}{dt}.\frac{dt}{dr} = \frac{L}{r^2}\frac{1}{\sqrt{2\mu[E - \frac{L^2}{2\mu r^2} - U(r)]}}.$$

The formal solution for the orbit is then obtained by integrating the previous expression

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$$\theta - \theta_0 = L \int_{r_0}^{r} \frac{dr}{r^2\sqrt{2\mu[E - \frac{L^2}{2\mu r^2} - U(r)]}}$$

where $\theta_0$ is a constant of integration. Inserting the expression of gravitational potential $U(r) = -\frac{C}{r}$ into this orbit equation we get

$$\theta - \theta_0 = L \int_{r_0}^{r} \frac{dr}{r\sqrt{2\mu E r^2 - L^2 + 2\mu C r]}}.$$

This integral over $r$ can be evaluated by converting it to a standard form, with the result

$$r = \frac{\frac{L^2}{\mu C}}{1 - \sqrt{1 + \frac{2EL^2}{\mu C^2}}\sin(\theta - \theta_0)}.$$

The usual conventions are to take $\theta_0 = -\frac{\pi}{2}$, so that the expression becomes

$$r = \frac{\frac{L^2}{\mu C}}{1 - \sqrt{1 + \frac{2EL^2}{\mu C^2}}\cos\theta}.$$

Introducing the parameters $r_0 = \frac{L^2}{\mu C}$ and $e = \sqrt{1 + \frac{2EL^2}{\mu C^2}}$, we obtain the equation as $r = \frac{r_0}{1 - e\cos\theta}$. Rewriting the equation in standard Cartesian form we get

$$(1 - e^2)x^2 - 2er_0 x + y^2 = {r_0}^2.$$

Mathematically, this equation describes the *conic sections*, such as hyperbola, parabola, ellipse, circle etc. traced out by a plane cutting a cone at various angles. Physically, $r_0$ is the radius of the circular orbit corresponding to the given values of $L$, $\mu$ and $C$. The dimensionless parameter $e$, known as the *eccentricity*, characterizes the shape of the conic section (or orbit).

**Shapes of the Orbit.** The various possibilities of the shapes of the orbits are discussed below.

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1. When $e > 1$, we get $E > 0$, which means that the system is unbounded. The coefficients of $x^2$ and $y^2$ are unequal and opposite in sign and the equation has the form $y^2 - Ax^2 - Bx =$ constant, which is the equation of a *hyperbola*.

2. When $e = 1$, hence $E = 0$, therefore the system is on the border between bounded and unbounded. The equation becomes $y^2 - Bx =$ constant, which is the equation of a *parabola*.

3. When $0 < e < 1$, we obtain $-\frac{\mu C^2}{2L^2} < E < 0$, which shows that the total energy $E$ is negative or the system is bounded. The coefficients of $x^2$ and $y^2$ are unequal but of the same sign now. The equation has the form $y^2 + Ax^2 - Bx =$ constant, which is the equation of an *ellipse*. The presence of the term linear in $x$ means that the geometric centre of the ellipse is not at the origin of coordinates (shown in Fig. 1).

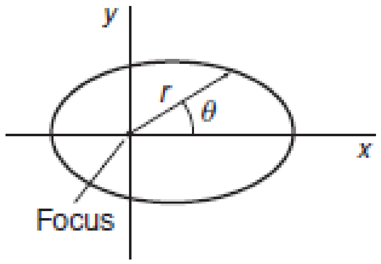


Fig. 1

4. When $e = 0$, $E$ has its lowest possible value $-\frac{\mu C^2}{2L^2}$. The system still remains bounded. The equation of the orbit becomes $x^2 + y^2 = {r_0}^2$. The ellipse degenerates to a *circle*, with equation $r = r_0 =$ constant.

## Elliptic Orbits for Planets:

Elliptic orbits are so important in astronomy and astrophysics that it is worth looking at their properties in more detail. For elliptic orbits, we just found that $E < 0$ and the eccentricity is $0 < e < 1$. In Cartesian coordinates the equation for an ellipse becomes

$$(1 - e^2)x^2 - 2er_0 x + y^2 = {r_0}^2$$

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and we see that the ellipse is symmetric about the $x$ axis (same for $+y$ and $-y$) in our chosen coordinate system. Further, the term linear in $x$ shows that the ellipse is displaced from the origin along the $x$ axis. As shown in Fig. 1, the maximum value of $r$, which occurs at $\theta = 0$, is

$$r_{max} = \frac{r_0}{1-e\cos 0} = \frac{r_0}{1-e}$$

and the minimum value of $r$, which occurs at $\theta = \pi$, is

$$r_{min} = \frac{r_0}{1-e\cos\pi} = \frac{r_0}{1+e}.$$

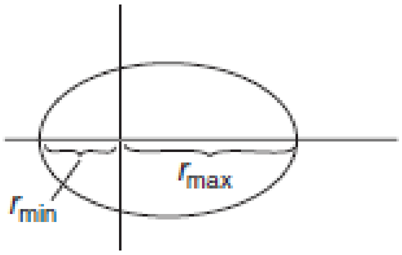


Fig. 2

As shown in Fig. 2, the length of the major axis is given by

$$A = r_{min} + r_{max} = r_0\left(\frac{1}{1+e} + \frac{1}{1-e}\right) = \frac{2r_0}{1-e^2}.$$

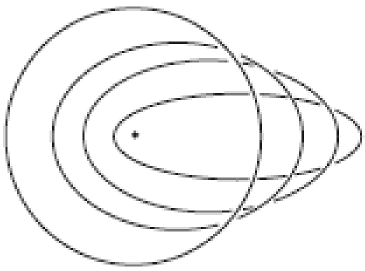

Fig. 3

Using the values of $r_0$ and $e$, defined before we get $A = \frac{2\frac{L^2}{\mu C}}{1-1-\frac{2EL^2}{\mu C^2}} = -\frac{C}{E}$. The length of the major axis is independent of $L$. Orbits with the same major axis

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have the same energy. For example, all the orbits shown in Fig. 3 correspond to the same value of $E$ although they have different values of $L$.

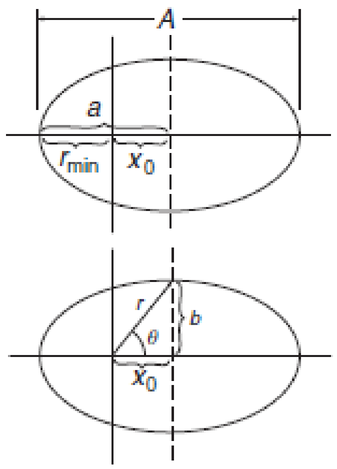


Fig. 4

The semi-major axis ($a$) is given by $a = \frac{A}{2} = \frac{r_0}{1-e^2} = -\frac{C}{2E}$ (as shown in Fig. 4). Also, the distance from the origin to the centre of the ellipse is given by

$$x_0 = a - r_{min} = \frac{r_0}{1-e^2} - \frac{r_0}{1+e} = \frac{r_0 e}{1-e^2}.$$

To find the length of the semi-minor axis $b = \sqrt{r^2 - x_0{}^2}$, we note that the tip of the semi-minor axis has angular coordinates given by $\cos\theta = \frac{x_0}{r}$. Hence,

$$r = \frac{r_0}{1-e\cos\theta} = \frac{r_0}{1-e\frac{x_0}{r}}$$

$$\text{or } r = r_0 + e x_0 = r_0\left(1 + \frac{e^2}{1-e^2}\right) = \frac{r_0}{1-e^2}.$$

Hence, $b = \sqrt{r^2 - x_0{}^2} = \frac{r_0}{1-e^2}\sqrt{1-e^2} = \frac{r_0}{\sqrt{1-e^2}} = \frac{L}{\sqrt{-2\mu E}}.$

**Proof of Kepler's 1st Law.** Now, we shall prove Kepler's 1st law. According to the definition of an ellipse, the sum of the distances from the two foci to a point on the ellipse is a constant. We start by assuming that one focus is at the origin, and by symmetry the other focus is therefore at $2x_0$, because the distance from the first focus to the centre of the ellipse is $x_0$. Let $r$ and $r'$ be the distances from the foci to a point on the ellipse, as shown in Fig. 5. We shall now show that $r + r' =$ constant, justifying our initial assumption.

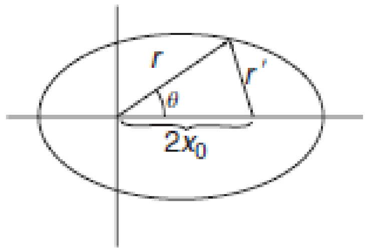


Fig. 5

By the law of cosines, $r'^2 = r^2 + 4{x_0}^2 - 4x_0 r\cos\theta = r^2 + 4{x_0}^2 - 4x_0\frac{r-r_0}{e}$. Using the expression $x_0$ of we get

$$r'^2 = r^2 - 2\frac{2r_0}{1-e^2}r + \frac{4{r_0}^2}{(1-e^2)^2} = \left(r - \frac{2r_0}{1-e^2}\right)^2$$

$$\text{or } r' = \pm\left(r - \frac{2r_0}{1-e^2}\right) = \pm(r - A).$$

Since $A > r$, we must choose the negative sign to keep $r' > 0$. Therefore,

$$r + r' = A = \text{constant}$$

which supports our assumption that a focus is at the origin and thus Kepler's 1st law is verified.

**Proof of Kepler's 2nd Law.** It is automatically verified since it was shown earlier that the areal velocity under any central force will remain constant.

**Proof of Kepler's 3rd Law.** In order to calculate the period of revolution ($T$), we start from the expression for angular momentum $L = \mu r^2\frac{d\theta}{dt}$. It can be arranged as

$$\frac{L}{2\mu}dt = \frac{1}{2}r^2 d\theta.$$

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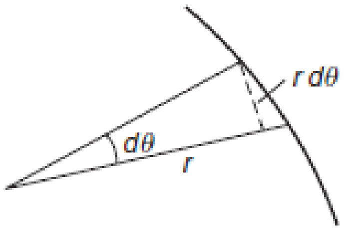


Fig. 6

But $\frac{1}{2}r^2d\theta = \frac{1}{2}(r)(rd\theta)$ is nothing but the area element in polar coordinates (shown in Fig. 6), so integrating the previous expression over a complete period $T$ sweeps out the area of the ellipse, which can be written as $\pi ab$.

$$\text{So, } \frac{L}{2\mu}T = \pi ab$$

$$\text{or } T = \frac{2\mu\pi ab}{L} = \frac{2\mu\pi\left(-\frac{C}{2E}\right)\left(\frac{L}{\sqrt{-2\mu E}}\right)}{L}$$

$$\text{or } T^2 = \frac{\mu^2\pi^2C^2}{E^2(-2\mu E)} = \frac{\pi^2\mu}{2C}\left(-\frac{C^3}{E^3}\right) = \frac{\pi^2\mu}{2C}A^3.$$

It proves Kepler's 3rd law, where $k = \frac{\pi^2\mu}{2C}$ is essentially the same for all planets about the sun. Despite variations of $\approx 100$ in the major axis and $\approx 1000$ in the period for various planets, the value of $\frac{T^2}{A^3}$ is constant to within 0.05% (illustrated in Table 1).

| Planet | $e$ | $A$ (km) | $T$ (s) | $\frac{T^2}{A^3}$ ($s^2km^{-3}$) |
|---|---|---|---|---|
| Earth | 0.017 | $2.99 \times 10^8$ | $3.16 \times 10^7$ | $3.74 \times 10^{-11}$ |
| Jupiter | 0.048 | $1.557 \times 10^9$ | $3.743 \times 10^8$ | $3.71 \times 10^{-11}$ |
| Mars | 0.093 | $4.56 \times 10^8$ | $5.93 \times 10^7$ | $3.71 \times 10^{-11}$ |
| Mercury | 0.206 | $1.16 \times 10^8$ | $7.62 \times 10^6$ | $3.72 \times 10^{-11}$ |

Table 1

**Geostationary Orbit and Geosynchronous Orbit.** For many communication purposes, satellites are typically placed in a circular *geosynchronous* orbit. If the orbit is in the equatorial plane of the earth, it is called *geostationary*. A satellite's orbital speed in a geostationary orbit is set to match the angular velocity $\omega_e$ of the rotating earth, so that as seen from the earth the satellite is stationary above a fixed point on the equator.

## Basics of Global Positioning System (GPS):

The Global Positioning System (GPS) is a space-based navigations system that provides location and time information in all weather conditions, anywhere on or near the earth where there is an unobstructed line of sight to four or more GPS satellites.

The GPS does not require the user to transmit any data, and it operates independently of any telephonic or internet reception, though these technologies can enhance the usefulness of the GPS positioning information. The GPS provides critical positioning capabilities to military, civil and commercial users around the world. The system is made freely accessible to anyone with a GPS receiver.

The GPS receiver calculates its own position and time based on data received from multiple GPS satellites. Each satellite carries an accurate record of its position and time, and transmits that data to the receiver. The satellites carry very stable atomic clocks that are synchronized with one another and with ground clocks. Any drift from time maintained on the ground is corrected daily. In the same manner, the satellite locations are known with great precision. GPS receivers have clocks as well, but they are less stable and less precise. Since the speed of radio waves is constant and independent of the satellite speed, the time delay between when the satellite transmits a signal and the receiver receives it is proportional to the distance from the satellite to the receiver. At a minimum, four satellites must be in view of the receiver for it to compute four unknown quantities (three position coordinates and clock deviation from satellite time).

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## Weightlessness:

The concept of *weightlessness* is one that is easily misunderstood by most people until they have been exposed to introductory Physics for the first time. For example, many of us learn early on that weight depends on force of gravity $F = mg$, and that we weigh more on earth than on the moon because the strength of gravity on earth ($g = 9.8$ ms$^{-2}$) is greater than that of the moon ($g = 1.6$ ms$^{-2}$). It is no surprise that one common misconception is that an object becomes weightless when the force of gravity becomes equal to zero. Strictly speaking, this statement would be true if not for the fact that the force of gravity can never be exactly equal to zero. This is because gravity has an effect even over incredibly long distances (or long range force), despite the fact that it is the weakest of the fundamental forces. So if the force of gravity is never equal to zero, the natural question arises about how anything can be considered weightless. In a very simple way, weightlessness can be more accurately defined as *the feeling of not having any weight*. This feeling is usually caused or enhanced by (a) the lack of a pushing or pulling force from nearby surfaces and/or (b) the perception that one's velocity relative to one's surroundings is equal to zero.

Both pushing and pulling forces and relative velocity are interrelated events that occur simultaneously to produce the feeling of weightlessness. Therefore, we can refine our earlier statement about the relative velocity of the object to its surroundings in the following way. The feeling of weightlessness is achieved when both the object and its surroundings experience the same net acceleration. This concept clearly explains why astronauts in the International Space Station or any other satellites or space shuttles are weightless. Intuitively, many people may say that there is no gravity in space or that the strength of gravity in orbit is much smaller. In fact, the opposite is true. Without gravity, satellites would not be able to orbit around the earth and would simply move in straight line into space. The reason why astronauts feel weightless is because they are accelerating toward the earth at the same rate as the shuttle they are in.

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(All the figures have been collected from the above mentioned references)

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