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Envelope

Definition: A curve which touches each member of a given family of curves is called envelope of that family. That is an **envelope** of a planar family of curves is a curve that is tangent to each member of the family at some point.

Procedure to find envelope for the given family of curves:

Case 1: Envelope of one parameter

Let us consider y=f(x) to be the given family of curves.

Step 1: Differentiate w.r.t to the parameter partially, and find the value of the parameter

Step 2**:** By substituting the value of parameter in the given family of curves, we get required envelope.

Special Case: If the given equation of curve is quadratic in terms of parameter, then envelope is given by discriminant =0.

Case 2: Envelope of two parameter

Let us consider y=f(x)to be the given family of curves, and a relation connecting these two parameters

Step 1: Obtain one parameter in terms of other parameter from the given relation

Step 2: Substitute in the given equation of curve, so that the problem of two parameter converts to problem of one parameter.

Step 3: Use one parameter technique to obtain envelope for the given family of curve

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Problems:

1. **Find the envelope of the family of straight line**

$$\boldsymbol{y = mx + \sqrt{a^2m^2 + b^2}},$$

m is the parameter.

Solution: Given equation of family of curves is $\boldsymbol{y = mx + \sqrt{a^2m^2 + b^2}}$

$$\Rightarrow y - mx = \sqrt{a^2m^2 + b^2}$$

$$(y - mx)^2 = a^2m^2 + b^2$$

$$y^2 - 2mxy + m^2x^2 = a^2m^2 + b^2$$

$$m^2(x^2 - a^2) - 2mxy + (y^2 - b^2) = 0.$$

Step 1:

Differentiate partially w.r.t the parameter m

$$2m(x^2 - a^2) - 2xy = 0$$

$$\Rightarrow m = \frac{xy}{(x^2 - a^2)}$$

Step 2:

Substitute the value of in the given family of curves

$$m^2(x^2 - a^2) - 2mxy + (y^2 - b^2) = 0$$

$$\Rightarrow \left(\frac{xy}{(x^2 - a^2)}\right)^2(x^2 - a^2) - \frac{2xy}{(x^2 - a^2)}xy + (y^2 - b^2) = 0$$

$$\Rightarrow \frac{x^2y^2}{(x^2 - a^2)} - \frac{2x^2y^2}{(x^2 - a^2)} + (y^2 - b^2) = 0$$

$$\Rightarrow -\frac{x^2y^2}{(x^2-a^2)} + (y^2-b^2) = 0$$

$$\Rightarrow x^2y^2 = (x^2-a^2)(y^2-b^2)$$

$$x^2b^2 + a^2y^2 = a^2b^2$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The envelope of the given family of straight lines is an ellipse.

2. **Find the envelope of family of straight line**

$$\boldsymbol{\frac{x}{a} + \frac{y}{b} = 1}$$

, where a, b **are two parameters which are connected by the relation** $\boldsymbol{a+b=c}$.

Solution: Given equation of family of straight lines is

$$\frac{x}{a} + \frac{y}{b} = 1 \ldots\ldots (1)$$

Also given,

$$a + b = c$$

$$\Rightarrow b = c - a \ldots\ldots (2)$$

Substituting (2) in (1), we get

$$\frac{x}{a} + \frac{y}{c-a} = 1$$

Step 1: Differentiate w.r.t a partially, we get

$$\frac{x}{-a^2} + \frac{y}{(c-a)^2} = 0$$

$$\Rightarrow \frac{x}{a^2} = \frac{y}{(c-a)^2}$$

$$\Rightarrow \frac{(c-a)^2}{a^2} = \frac{y}{x}$$

$$\Rightarrow \frac{c-a}{a} = \sqrt{\frac{y}{x}}$$

$$\Rightarrow \frac{c}{a} = 1 + \sqrt{\frac{y}{x}}$$
$$\Rightarrow \frac{c}{a} = \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}}$$
$$\Rightarrow a = \frac{c\sqrt{x}}{\sqrt{x}+\sqrt{y}}$$

Now, substitute the value of a in b=c-a

$$b = c - \frac{c\sqrt{x}}{\sqrt{x}+\sqrt{y}}$$

$$\Rightarrow b = \frac{c\sqrt{y}}{\sqrt{x}+\sqrt{y}}$$

Step 2: Substitute the values of a and b in the given family of curves

$$\frac{x}{a} + \frac{y}{b} = 1$$

we get

$$\frac{x}{\frac{c\sqrt{x}}{\sqrt{x}+\sqrt{y}}} + \frac{y}{\frac{c\sqrt{y}}{\sqrt{x}+\sqrt{y}}} = 1$$

$$\Rightarrow \frac{x(\sqrt{x}+\sqrt{y})}{c\sqrt{x}} + \frac{y(\sqrt{x}+\sqrt{y})}{c\sqrt{y}} = 1$$

$$\Rightarrow (\sqrt{x}+\sqrt{y})^2 = c$$

$$\Rightarrow \sqrt{x}+\sqrt{y} = \sqrt{c}.$$

This is the required envelope.

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Exercise:

Find the envelopes for the family of curves

1. $y = e^{-x}\sin(x + c)$
2. $y = 1 + (\frac{1}{4}x^2 + c)^2$
3. $y = \sin\left(\frac{1}{2}x^2 + c\right)$.