Dr. Avradip Pradhan,
Assistant Professor,
Department of Physics,
Narajole Raj College, Narajole.

C2T (Mechanics)

Topic – Gravitation and Central Force Motion (Part – 1)

Introduction:

Gravitation or just gravity, the most familiar of the four fundamental forces, played a crucial role in the development of the so-called classical mechanics. It is the force of attraction between any two bodies. All the objects in the universe attract each other with a certain amount of force, but in most of the cases, the force is too weak to be observed due to the very large distance of separation. Besides, gravity's range is infinite but the effect becomes weaker as objects move away. Sir Isaac Newton discovered the law of universal gravitation in 1666, the same year that he formulated his laws of motion. By calculating the motion of two gravitating particles, Newton was able to derive Kepler's empirical laws of *planetary motion*.

According to Newton's law of gravitation, two particles attract each other with a force that is proportional to the product of their masses and inversely proportional to the square of the distance between them. It is important to note that, force of gravitation is always *attractive*. Gravitation can generally exist in two main instances, (a) gravitation may be the attraction of objects by the earth and (b) gravitation may be the attraction of objects in outer space.

Newton's Law of Gravitation:

This verbal description of the gravitational force is essentially correct but not useful for solving problems, for which we need a mathematical expression. Let us consider two point-size particles, a and b, with masses M_a and M_b respectively, separated by a distance r (as shown in Fig. 1). Let $\vec{F}_{ba}$ be the force exerted on particle b by particle a. Our verbal description of the magnitude of the gravitational force can be expressed mathematically as

$$\left|\vec{F}_{ba}\right| = F_{ba} \propto \frac{M_a M_b}{r^2}$$

Dr. Avradip Pradhan,
Assistant Professor,
Department of Physics,
Narajole Raj College, Narajole.

$$\text{or } F_{ba} = G\frac{M_a M_b}{r^2}$$

G is known as the *universal gravitational constant* or *Newtonian constant*. It is defined as the force of attraction between two unit masses separated by unit distance. The value of G can be found by measuring the force between masses in a known geometry. The mass of the earth can also be found from G, the acceleration due to gravity g, and the radius of the earth R_e.

Dimension of G. The dimension of G is calculated by a dimensional analysis of the previous equation. Since, $F_{ba} = G\frac{M_a M_b}{r^2}$ or $G = \frac{F_{ba} r^2}{M_a M_b}$, we can write

$$[G] = \frac{[F_{ba}][r^2]}{[M_a][M_b]} = \frac{[MLT^{-2}][L^2]}{[M][M]} = [L^3 M^{-1} T^{-2}].$$

The value of G is 6.673×10^{-11} $m^3 kg^{-1} s^{-2}$. G is difficult to measure because of the weakness of gravity, and at a relative uncertainty of 10^{-4}, it is the least accurately known of the fundamental constants in Physics.

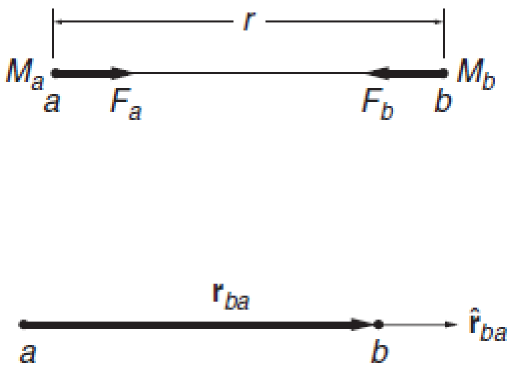

Fig. 1

Vector Notation. As we will see later, the gravitational force between two particles is a *central force* because it is directed along the line joining them. Vector notation is ideally suited for describing these properties mathematically. By convention, we introduce the vector $\vec{r}_{ba}$ that extends from the particle exerting the force, particle *a* in this case, to the particle experiencing the force, particle b. It is obvious that $\vec{r}_{ab} = -\vec{r}_{ba}$ and $|\vec{r}_{ba}| = |\vec{r}_{ab}| = r$. Introducing the unit vector $\hat{r}_{ba} = \vec{r}_{ba}/r$, we have

$$\vec{F}_{ba} = -G\frac{M_a M_b}{r^2}\hat{r}_{ba}$$

Here the negative sign indicates that the force on particle b is directed toward particle a, that is, the force is *attractive* in nature. Similarly, the force on a due to b is (using the fact $\hat{r}_{ab} = -\hat{r}_{ba}$)

$$\vec{F}_{ab} = -G\frac{M_a M_b}{r^2}\hat{r}_{ab} = G\frac{M_a M_b}{r^2}\hat{r}_{ba} = -\vec{F}_{ba}$$

Therefore the forces on the two particles are equal and opposite, as Newton's third law requires.

Gravitational Field, Gravitational Potential Energy, Gravitational Potential:

Gravitational Field. By the gravitational field (or *attraction*) of a particle or a system of particles at a point, we mean the force of attraction experienced by a particle of unit mass ($m = 1$) placed at that point. If f is the magnitude of the gravitational field due to the source particle or object of mass M, then at a point r from the source particle we can write

$$f = G\frac{M}{r^2}.$$

Gravitational field is a vector quantity, the direction of which will be the same as that of the gravitational force at each point in space.

Gravitational Potential Energy & Gravitational Potential. In a gravitational field, moving a mass (say, m) from one point A to another point B requires an expenditure of work. This amount of work done may be negative or positive according as the body is moved in the direction or against the direction of the force of attraction. This amount of work done is a direct measure of the gravitational potential energy difference (dU) between the two points A and B between which the mass is moved. Gravitational potential (V) is basically same

Dr. Avradip Pradhan,
Assistant Professor,
Department of Physics,
Narajole Raj College, Narajole.

as gravitational potential energy (U), where the moving mass is considered to be of unit magnitude ($m = 1$).

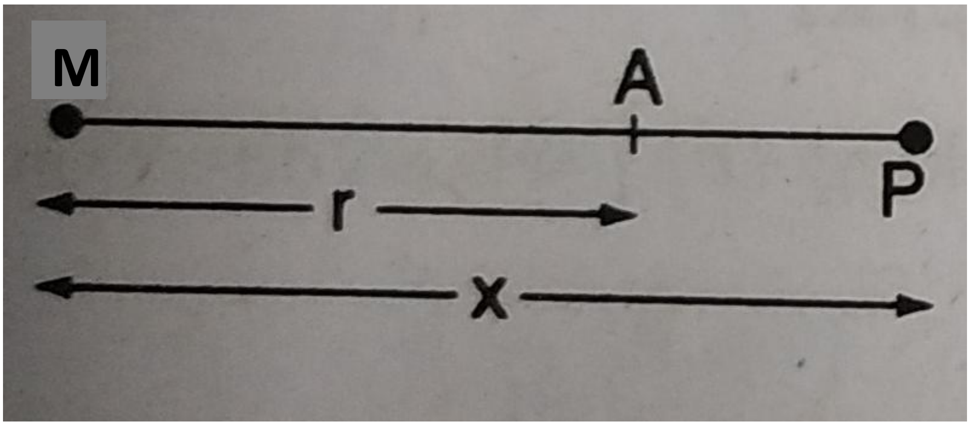

Fig. 2

If one of the two points, say A be at infinity, we get the absolute value of the potential energy U (or potential V) at the other point B. This is because, the gravitational potential at the point A situated at infinity is considered to be zero. Thus the gravitational potential at a point in a gravitational field is measured by negative of the amount of work done in bringing a unit mass from infinity to that point. Let the gravitational field be due to a source particle of mass M. We are to calculate the gravitational potential at A which is a distant r from the particle. We consider a unit mass located at a distance x from the particle (see Fig. 2). Now the force of attraction on the unit mass is $-G\frac{M}{x^2}\hat{x}$ (where $\hat{x}$ is pointing away from the mass M) and the work done in moving the unit mass through a distance dx (vectorially, $dx\hat{x}$) in the direction of the force will be

$$dW = -G\frac{M}{x^2}dx$$

Hence, the total work done in moving the unit mass from infinity to the point A is given by $W = \int dW = -GM\int_{\infty}^{r}\frac{dx}{x^2} = G\frac{M}{r}$. According to the definition of the gravitational potential, the value of this at the point A is given by

$$V = -W = -G\frac{M}{r}.$$

The negative sign clearly indicates that the potential is *attractive* in nature.

Dr. Avradip Pradhan,
Assistant Professor,
Department of Physics,
Narajole Raj College, Narajole.

Calculation of Gravitational Potential for Extended Spherical Bodies:

(1) Thin Spherical Shell. We consider a thin spherical shell of radius R and surface density σ (shown in Fig. 3). Mass of the whole shell (M) is given by

$$M = 4\pi R^2 \sigma.$$

Let P be a point at a distance r from the centre O of the shell. We consider a slice $BB'CC'$ in the form of a ring perpendicular to OP. The slice is at an angle θ with OP. The area of such a slice is given by

$$dS = 2\pi R \sin\theta \, R d\theta = 2\pi R^2 \sin\theta \, d\theta.$$

So, the mass of the slice is $dM = \sigma dS = 2\pi\sigma R^2 \sin\theta \, d\theta$.

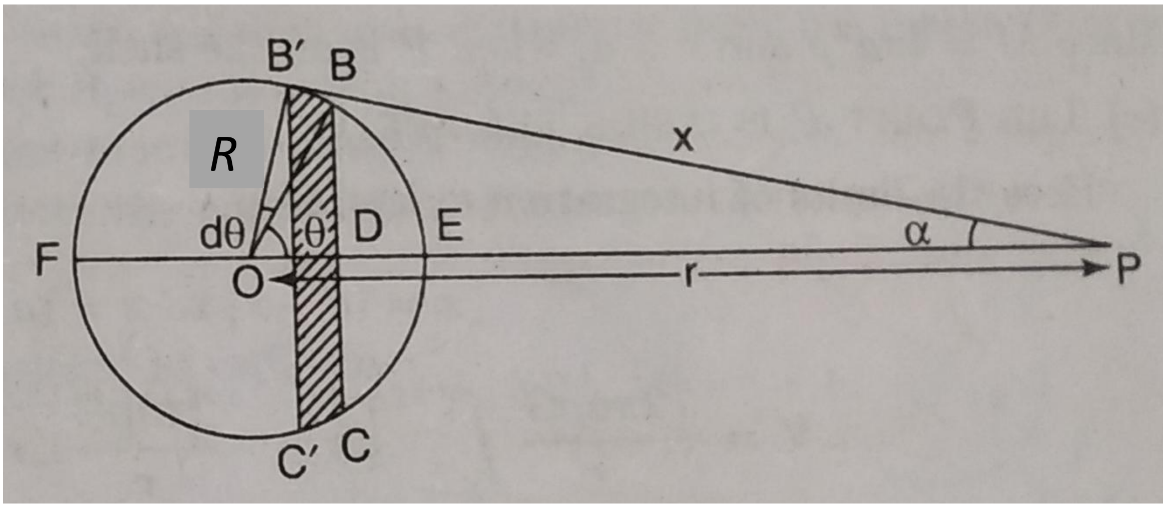

Fig. 3

It is important to note that every point on the slice is situated at equal distance x from the point P. Hence, the potential at P due to this slice

$$dV = -G\frac{dM}{x} = -G\frac{2\pi\sigma R^2 \sin\theta d\theta}{x}.$$

Now from the triangle BOP, we can write

$$x^2 = R^2 + r^2 - 2Rr\cos\theta$$

Dr. Avradip Pradhan,
Assistant Professor,
Department of Physics,
Narajole Raj College, Narajole.

Differentiating we have $2xdx = 2Rr\sin\theta\, d\theta$ or $\sin\theta\, d\theta = \frac{xdx}{Rr}$. Substituting the value of $\sin\theta\, d\theta$, we have $dV = -G\frac{2\pi\sigma R dx}{r}$. Hence, the potential for the whole shell is obtained by integrating the above expression and is given by

$$V = -G\frac{2\pi\sigma R}{r}\int dx$$

Now, we consider the following cases depending on the position of the point P.

(a) The Point is outside the Shell. In this case, $r > R$ and the limits of integration extend from $x = r - R$ to $x = r + R$. Hence,

$$V = -G\frac{2\pi\sigma R}{r}\int_{r-R}^{r+R} dx = -G\frac{4\pi R^2\sigma}{r} = -G\frac{M}{r}.$$

where $M = 4\pi R^2\sigma$, is the mass of the shell. This is the value of the potential that a point mass M situated at O would have produced at P. That is, the shell behaves as if its whole mass were concentrated at its centre.

(b) The Point is on the surface of the Shell. In this case, $r = R$ and the limits of integration extend from $x = 0$ to $x = 2R$. Hence,

$$V = -G\frac{2\pi\sigma R}{R}\int_{0}^{2R} dx = -G\frac{4\pi R^2\sigma}{R} = -G\frac{M}{R}.$$

where $M = 4\pi R^2\sigma$, is again the mass of the entire shell.

(c) The Point is inside the Shell. In this case, $r < R$ and the limits of integration extend from $x = R - r$ to $x = R + r$. Hence, we can write

$$V = -G\frac{2\pi\sigma R}{r}\int_{R-r}^{R+r} dx = -G4\pi R\sigma = -G\frac{M}{R}.$$

Thus the potential at any point inside a shell is constant and is equal to that on the surface of the shell.

(2) Uniform Solid Sphere. A solid sphere of radius R may be supposed to be made up of a large number of thin uniform concentric spherical shells (as shown in Fig. 4), the masses of which will be concentrated at the centre O.

Dr. Avradip Pradhan,
Assistant Professor,
Department of Physics,
Narajole Raj College, Narajole.

(a) The Point is outside the Sphere. Here the potential due to all such shells at an external point P is given by

$$V = -G\frac{m_1}{r} - G\frac{m_2}{r} - G\frac{m_3}{r} - G\frac{m_4}{r} - \cdots$$

where m_1, m_2, m_3...etc are the masses of the individual shells under consideration. Now, mass of the entire sphere would be

$$M = m_1 + m_2 + m_3 + \cdots$$

Therefore, we can write $V = -G\frac{(m_1+m_2+m_3+\cdots)}{r} = -G\frac{M}{r}$.

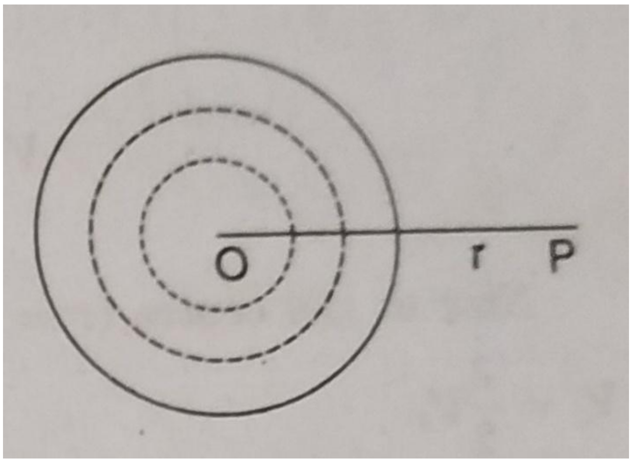

Fig. 4

(b) The Point is on the surface of the Sphere. The calculation may be done exactly in a similar way and it can be shown that the potential is given by

$$V = -G\frac{M}{R}$$

where R is the radius of the solid sphere.

(c) The Point is inside the Sphere. At a point inside the sphere such as P, which is at a distance r from the centre O, it is required to find the potential. The point P, as shown in Fig. 5, is external to a solid sphere of radius r and internal to a thick shell of radii R (external) and r (internal). Let V_1 and V_2 denote the potentials at P due these two parts. So, we can write

Dr. Avradip Pradhan,
Assistant Professor,
Department of Physics,
Narajole Raj College, Narajole.

$$V_1 = -G\frac{\frac{4}{3}\pi r^3\rho}{r} = -\frac{4}{3}\pi G\rho r^2$$

where ρ is the mass density of the sphere.

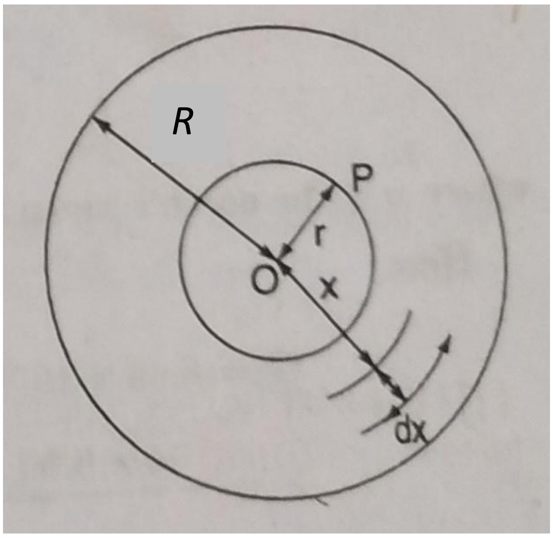

Fig. 5

To calculate V_2 we consider a thin concentric spherical shell of radius x and thickness dx, of which P is an internal point. Therefore, the potential at P due to the thin spherical shell under consideration is given by

$$dV_2 = -G\frac{4\pi x^2 dx\rho}{x} = -G4\pi\rho x dx.$$

Hence, for the thick shell of radii R and r, the potential is given by

$$V_2 = -G4\pi\rho\int_r^R x dx = -2\pi G\rho(R^2 - r^2).$$

Hence, the potential at P due to the whole solid sphere is given by

$$V(r) = V_1 + V_2 = -\frac{4}{3}\pi G\rho r^2 - 2\pi G\rho(R^2 - r^2)$$

$$\text{or } V = -\frac{2}{3}\pi G\rho(3R^2 - r^2)$$

As the mass of the whole sphere is given by $M = \frac{4}{3}\pi R^3\rho$, so we obtain

$$V = -\frac{GM}{2R^3}(3R^2 - r^2).$$

Now at the centre ($r = 0$), $V = -\frac{3}{2}G\frac{M}{R}$ and at the surface ($r = R$), $V = -G\frac{M}{R}$.

Calculation of Gravitational Field for Extended Spherical Bodies:

(1) Thin Spherical Shell. With reference to the Fig. 3, the gravitational field or attraction at P due to the slice $BB'CC'$ is given by $G\frac{dM}{x^2} = G\frac{2\pi\sigma R^2 \sin\theta d\theta}{x^2}$ directed along PB. The resultant field will be directed along PO and the magnitude is given by

$$df = G\frac{2\pi\sigma R^2 \sin\theta d\theta}{x^2}\cos\alpha = G\frac{2\pi\sigma R^2 \sin\theta d\theta}{x^2}\frac{r-R\cos\theta}{x}$$

Again from the triangle BOP, we can write $x^2 = R^2 + r^2 - 2Rr\cos\theta$.

Differentiating we have $2xdx = 2Rr\sin\theta\, d\theta$ or $\sin\theta\, d\theta = \frac{xdx}{Rr}$. Substituting the value of $\sin\theta\, d\theta$, we get

$$df = G\frac{2\pi\sigma R^2}{x^3}\frac{xdx}{Rr}\frac{x^2-R^2+r^2}{2r} = G\frac{\pi R\rho}{r^2}\left[1 + \frac{r^2-R^2}{x^2}\right]dx.$$

Hence the field for the whole shell is obtained by integrating the above expression. We now consider the following cases depending on the position of the point P.

(a) The Point is outside the Shell. In this case, $r > R$ and the limits of integration extend from $x = r - R$ to $x = r + R$. Hence, we obtain

$$f = G\frac{\pi R\rho}{r^2}\int_{r-R}^{r+R}\left[1 + \frac{r^2-R^2}{x^2}\right]dx = G\frac{4\pi R^2\sigma}{r^2}$$

$$\text{or } f = G\frac{M}{r^2}$$

$$\text{or } \vec{f} = -G\frac{M}{r^2}\hat{r}, \text{ vectorially.}$$

where M is the mass of the entire shell. Thus the shell attracts an external particle as if its mass were concentrated at its centre.

Dr. Avradip Pradhan,
Assistant Professor,
Department of Physics,
Narajole Raj College, Narajole.

(b) The Point is on the surface of the Shell. In this case, $r = R$ and the limits of integration extend from $x = 0$ to $x = 2R$. Hence,

$$f = G\frac{\pi R\rho}{R^2}\int_0^{2R}\left[1 + \frac{r^2 - R^2}{x^2}\right]dx = G4\pi\sigma = G\frac{M}{R^2}$$

$$\text{or } \vec{f} = -G\frac{M}{R^2}\hat{r}, \text{ vectorially.}$$

(c) The Point is inside the Shell. In this case, $r < R$ and the limits of integration extend from $x = R - r$ to $x = R + r$. Hence, we can write

$$f = G\frac{\pi R\rho}{r^2}\int_{R-r}^{R+r}\left[1 + \frac{r^2 - R^2}{x^2}\right]dx = G\frac{\pi R\sigma}{r^2}(2r - 2r) = 0.$$

Thus, there is no resultant gravitational field at any point inside the shell.

(2) Uniform Solid Sphere. As explained before, the field due to a solid sphere may be obtained by dividing the sphere into a number of thin uniform concentric shells, the masses of which may be supposed to be concentrated at the centre O (as shown in Fig. 4).

(a) The Point is outside the Sphere. The resultant field due to all such shells is equal to the intensity due to the sphere directed along PO and is given by

$$f = G\frac{m_1}{r^2} + G\frac{m_2}{r^2} + G\frac{m_3}{r^2} + G\frac{m_4}{r^2} + \cdots$$

where m_1, m_2, m_3…etc are the masses of the constituent shells. Therefore,

$$f = G\frac{(m_1 + m_2 + m_3 + \cdots)}{r^2} = G\frac{M}{r^2} \text{ (vectorially } \vec{f} = -G\frac{M}{r^2}\hat{r})$$

where $M = m_1 + m_2 + m_3 + \cdots$ is the mass of the solid sphere.

(b) The Point is on the surface of the Sphere. The calculation may be done exactly in a similar way and it can be shown that the field is given by

$$f = G\frac{M}{R^2} \text{ (vectorially } \vec{f} = -G\frac{M}{R^2}\hat{r})$$

where R is the radius of the solid sphere.

(c) The Point is inside the Sphere. At a point P inside the sphere which is at a distance r from the centre O, it is required to the gravitational potential (as shown in Fig. 5). In this case, the point P is external to a solid sphere of radius r and internal to a thick shell of radii R (external) and r (internal).

Here the thick shell does not contribute anything towards the gravitational field at its internal point P (according to our calculation).

Hence the resultant field at P is only due to the solid sphere of radius r having mass $= \frac{4}{3}\pi r^3 \rho$, where ρ is the mass density of the sphere. Mass of the entire sphere M can be related as $M = \frac{4}{3}\pi R^3 \rho$. So,

$$f = G\frac{\frac{4}{3}\pi r^3 \rho}{r^2} = \frac{4}{3}\pi G R \rho$$

$$\text{or } f(r) = G\frac{Mr}{R^3}$$

$$\text{or } \vec{f} = -G\frac{Mr}{R^3}\hat{r}, \text{ vectorially.}$$

Thus, inside the sphere, the field magnitude is directly proportional to the distance from the centre.

Acceleration Due to Gravity:

At the surface of the earth, the gravitational force on mass m is given by

$$\vec{F} = -G\frac{M_e m}{{R_e}^2}\hat{r}$$

where M_e and R_e are the mass and radius of the earth respectively. The acceleration due to earth's gravitation, known as *acceleration due to gravity* is

$$\vec{a} = \frac{\vec{F}}{m} = -G\frac{M_e}{{R_e}^2}\hat{r}$$

As we expect, this acceleration is independent of m. The acceleration due to the earth's gravity is universally designated by g. When g is written as a vector, the vector is directed *down* toward the centre of the earth given as

Dr. Avradip Pradhan,
Assistant Professor,
Department of Physics,
Narajole Raj College, Narajole.

$$\vec{g} = -G\frac{M_e}{{R_e}^2}\hat{r}$$

The value of g varies slightly over the surface of the earth, but if high accuracy is not required it can be taken to have a nominal value of 9.8 ms^{-2} = 980 cms^{-2} ≈ 32 fts^{-2}.

This concludes part 1 of this e-report.

The discussion will be continuing in the part 2 of this e-report.

Reference(s):

An Introduction to Mechanics, Kleppner & Kolenkow, Cambridge University Press

A Treatise on General Properties of Matter, Chatterjee & Sengupta, New Central Book Agency

(All the figures have been collected from the above mentioned references)