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C1T (Mathematical Physics – I)

Topic – Vector Calculus (Part – 1)

Introduction:

The simplest kind of physical quantity is one that can be completely specified just by a number, called its *magnitude* together with the units in which it is measured. Such a quantity is called a *scalar* and examples include temperature, time, energy etc. A *vector* is a quantity that requires both a magnitude (≥ 0) and a direction in space to specify it completely. So, we may think of a vector as an arrow in space. A familiar example is force, which has a magnitude (strength) measured in Newtons (N) and a direction of application. The large number of vectors that are used to describe the physical world include velocity, displacement, momentum, electric field etc. Vectors are also used to describe quantities such as angular momentum and surface elements (a surface element has an area and a direction defined by the normal to its tangent plane). In such cases their definitions may seem somewhat arbitrary (though in fact they are standard) and not as physically intuitive as for usual vectors such as force. A vector is represented by an alphabet with an arrow over it. It can also be denoted by bold type alphabet, the convention of many textbooks.

Basic Properties of Vectors:

In this e-report, we consider basic vector algebra and illustrate just how powerful vector analysis can be. All the techniques are presented for three-dimensional space but most can be readily extended to more dimensions. We now discuss some basic and useful properties of the vectors.

Vector Addition and Vector Subtraction:

The resultant or vector sum of two vectors is the vector that results from performing first one and then the other one, as shown in Fig. 1. This process is known as *vector addition*. However, the principle of addition has physical

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meaning for vector quantities other than displacements, for example, if two forces act on the same body then the resultant force acting on the body is the vector sum of the two. The addition of vectors only makes physical sense if they are of a like kind, for example if they are both forces acting in three dimensions. It may be seen from Fig. 1 that vector addition is *commutative*, i.e.

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

for two vectors $\vec{A}$ and $\vec{B}$. It clearly shows the sum doesn't depend on the sequence in which the vectors are added.

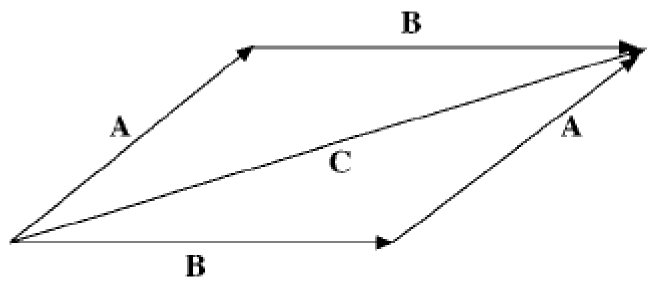

Fig. 1

The generalization of this procedure to the addition of three (or more) vectors is clear and leads to the *associative* property of addition (see Fig. 2), e.g.

$$\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$$

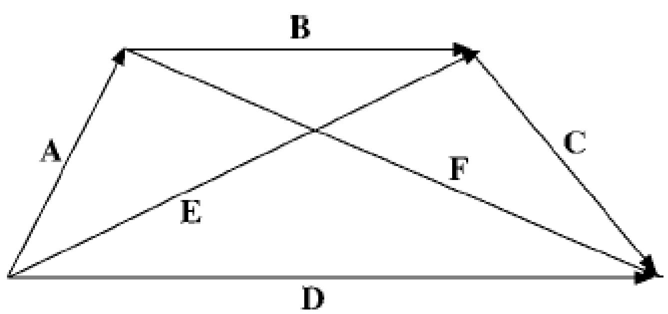

Fig. 2

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Thus, it is immaterial in what order any number of vectors are added.

Subtraction may be handled by defining the *negative* of a vector as a vector of the same magnitude but with reversed direction (as shown in Fig. 3). Then we define the subtraction as

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$

where $-\vec{B}$ is the negative vector of the vector $\vec{B}$. It is worth mentioning that the subtraction of two equal vectors yields the *zero* or *null vector*, denoted as $\vec{0}$, which has zero magnitude and no associated direction. Therefore, we get

$$\vec{A} - \vec{A} = \vec{0}$$

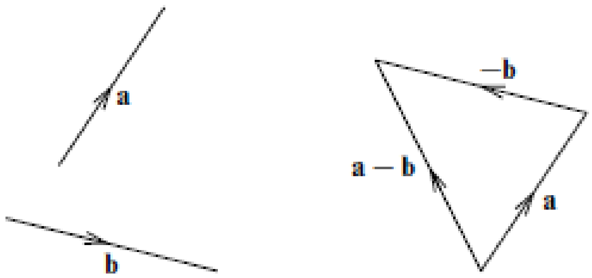

Fig. 3

Multiplication by a Scalar. Multiplication of a vector by a scalar (not to be confused with the scalar product, to be discussed later in another e-report) gives a vector in the same direction as the original but of a proportional magnitude. This can be seen in Fig. 4. The scalar may be positive, negative or zero. It can also be complex in some applications. Clearly, when the scalar is negative we obtain a vector pointing in the opposite direction to the original vector. Multiplication by a scalar is associative, commutative and distributive over addition. These properties may be summarized for arbitrary vectors $\vec{A}$ and $\vec{B}$ and arbitrary scalars α and β by

$$(\alpha\beta)\vec{A} = \alpha(\beta\vec{A}) = \beta(\alpha\vec{A})$$

$$(\alpha + \beta)\vec{A} = \alpha\vec{A} + \beta\vec{A}$$

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$$\alpha(\vec{A} + \vec{B}) = \alpha\vec{A} + \alpha\vec{B}$$

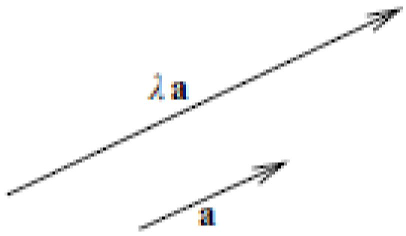

Fig. 4

Basis Vectors and Components:

Given any three different vectors $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$, which do not all lie in a plane, it is possible in a three dimensional space, to write any other vector in terms of scalar multiples of them as

$$\vec{A} = A_1\vec{e}_1 + A_2\vec{e}_2 + A_3\vec{e}_3 = \sum_{i=1}^{3} A_i\,\vec{e}_i$$

The three vectors $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$ are then said to form a *basis* (for the three dimensional space). The scalars A_1, A_2 and A_3, which may be positive, negative or zero, are called the *components* or *projections* of the vector $\vec{A}$ with respect to this basis. We say that the vector has been resolved into components.

Most often we shall use basis vectors that are mutually *perpendicular*, for ease of manipulation, though this is not necessary. In general, a basis set must obey the following properties.

(a) It must have as many basis vectors as the number of dimensions (in more formal language, the basis vectors must span the space) and

(b) It should be such that no basis vector may be described as a sum of the others, or the basis vectors must be linearly independent. Putting this mathematically, in N dimensions, we require

$$c_1\vec{e}_1 + c_2\vec{e}_2 + c_3\vec{e}_3 + \cdots + c_N\vec{e}_N = \sum_{i=1}^{N} c_i\,\vec{e}_i \neq \vec{0}$$

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for any set of coefficients $c_1, c_2, \ldots, c_N$ except $c_1 = c_2 = c_3 = \cdots = c_N = 0$.

In this e-report, we will only consider vectors in three dimensions; higher dimensionality can be achieved by simple extension. If we wish to label points in space using a Cartesian coordinate system (x, y, z), we may introduce the *unit vectors* (defined later) $\hat{x}$, $\hat{y}$ and $\hat{z}$, which point along the positive x, y and z axes respectively. A vector $\vec{A}$ may then be written as a sum of three vectors, each parallel to a different coordinate axis as

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$

A vector in three-dimensional space thus requires three components to describe fully both its direction and its magnitude. A displacement in space may be thought of as the sum of displacements along the x, y and z directions (as shown in Fig. 5). For brevity, the components of a vector $\vec{A}$ with respect to a particular coordinate system are sometimes written in the form (A_x, A_y, A_z). It is important to note that the basis vectors $\hat{x}$, $\hat{y}$ and $\hat{z}$ may themselves be represented by (1, 0, 0), (0, 1, 0) and (0, 0, 1) respectively.

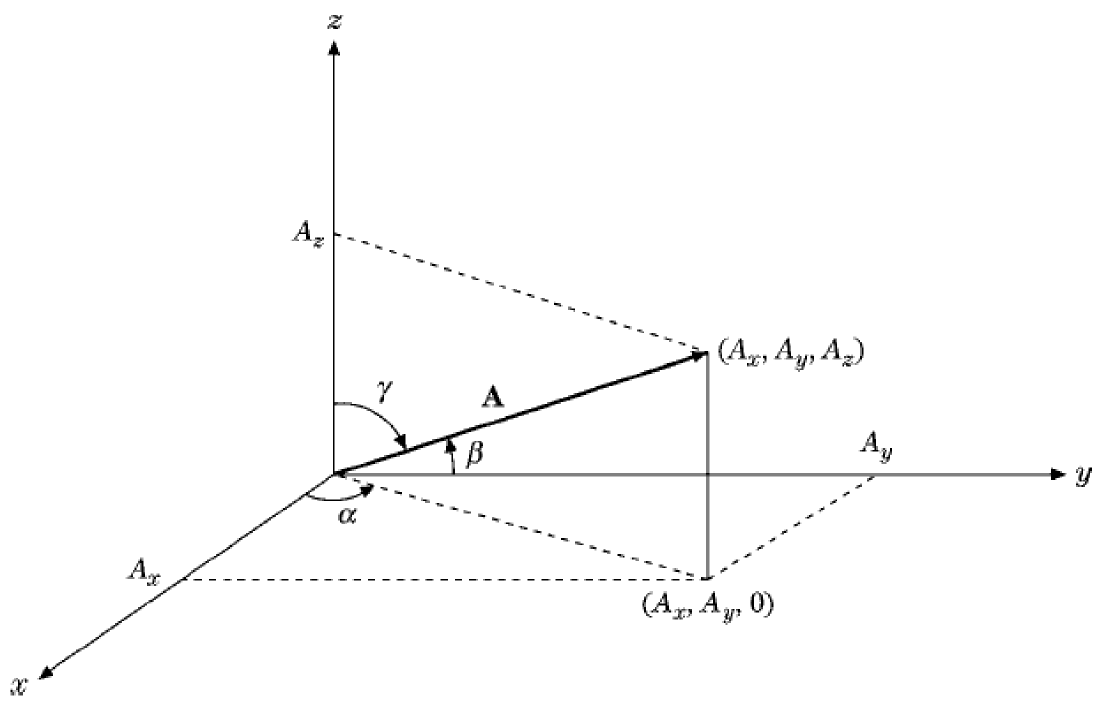

Fig. 5

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The representation of vector $\vec{A}$ by an arrow suggests an excellent visualization. Arrow $\vec{A}$ (shown in Fig. 5), starting from the origin, terminates at the end point (A_x, A_y, A_z). Thus, if we agree that the vector is to start at the origin, the positive end may be specified by giving the Cartesian coordinates (A_x, A_y, A_z) of the arrowhead. If a vector $\vec{A}$ vanishes, all of its components must vanish individually, that is, if $\vec{A} = \vec{0}$ then $A_x = A_y = A_z = 0$.

We can consider the addition and subtraction of vectors in terms of their components. The sum of two vectors $\vec{A}$ and $\vec{B}$ is found by simply adding their components, i.e.

$$\vec{A} + \vec{B} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z} + B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$$

$$= (A_x + B_x)\hat{x} + (A_y + B_y)\hat{y} + (A_z + B_z)\hat{z}$$

and their difference by subtracting them, as

$$\vec{A} - \vec{B} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z} - B_x\hat{x} - B_y\hat{y} - B_z\hat{z}$$

$$= (A_x - B_x)\hat{x} + (A_y - B_y)\hat{y} + (A_z - B_z)\hat{z}$$

Direction Cosines. Using A for the magnitude (will be defined next) of vector $\vec{A}$, we find that Fig. 5 shows that the endpoint coordinates and the magnitude are related by the following relations

$$A_x = A\cos\alpha, A_y = A\cos\beta, A_z = A\cos\gamma.$$

Here α, β and γ are the angles between the given vector and the positive x, y and z axes. With this $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the *direction cosines* of the given vector $\vec{A}$.

Magnitude of a Vector. The magnitude of the vector $\vec{A}$ is denoted by $|\vec{A}|$ or A. In terms of its components in three-dimensional Cartesian coordinates, the magnitude of $\vec{A}$ is given by

$$|\vec{A}| = A = \sqrt{{A_x}^2 + {A_y}^2 + {A_z}^2}\ .$$

$$\text{So, } |\vec{A}|^2 = A^2 = {A_x}^2 + {A_y}^2 + {A_z}^2.$$

Hence, the magnitude of a vector is a measure of its *length*. Such an analogy is useful for displacement vectors but magnitude is better described, by strength for vectors such as force or by speed for velocity vectors. From the relation $A^2 = {A_x}^2 + {A_y}^2 + {A_z}^2$ we also obtain a very important connection between the direction cosines as

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

Unit Vector. A vector whose magnitude equals unity is called a unit vector. The unit vector in the direction $\vec{A}$ is usually notated $\hat{A}$ and may be evaluated as

$$\hat{A} = \frac{\vec{A}}{A}.$$

The unit vector is a useful concept because a vector written as $\alpha\hat{A}$ has magnitude α and direction $\hat{A}$. Thus magnitude and direction are explicitly separated.

Transformation of Coordinate Axes:

In the preceding part vectors were defined or represented in two equivalent ways: (1) geometrically by specifying magnitude and direction, as with an arrow, and (2) algebraically by specifying the components relative to Cartesian coordinate axes. The second definition is adequate for the vector analysis. In this section two more refined, sophisticated and powerful definitions are presented. First, the vector field is defined in terms of the behaviour of its components under transformation (particularly, rotation) of the coordinate axes. This transformation theory approach leads into the tensor analysis and groups of transformations. Second, the component definition is refined and generalized according to the Mathematician's concepts of vector and vector space. This approach leads to function spaces, including the *Hilbert Space*. The definition of vector as a quantity with magnitude and direction is thus incomplete. On the

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one hand, we encounter quantities, such as elastic constants and index of refraction in anisotropic crystals, that have magnitude and direction but that are not vectors. On the other hand, our naive approach is awkward to generalize to extend to more complex quantities. We seek a new definition of vector field using our coordinate vector $\vec{r}$ as a prototype.

There is a physical basis for our development of a new definition. We describe our physical world by Mathematics, but it and any physical predictions we may make must be independent of our mathematical conventions. The physical system being analyzed or the physical law being enunciated cannot and must not depend on our choice or orientation of the coordinate axes. Specifically, if a quantity does not depend on the orientation of the coordinate axes, it is called a *scalar*.

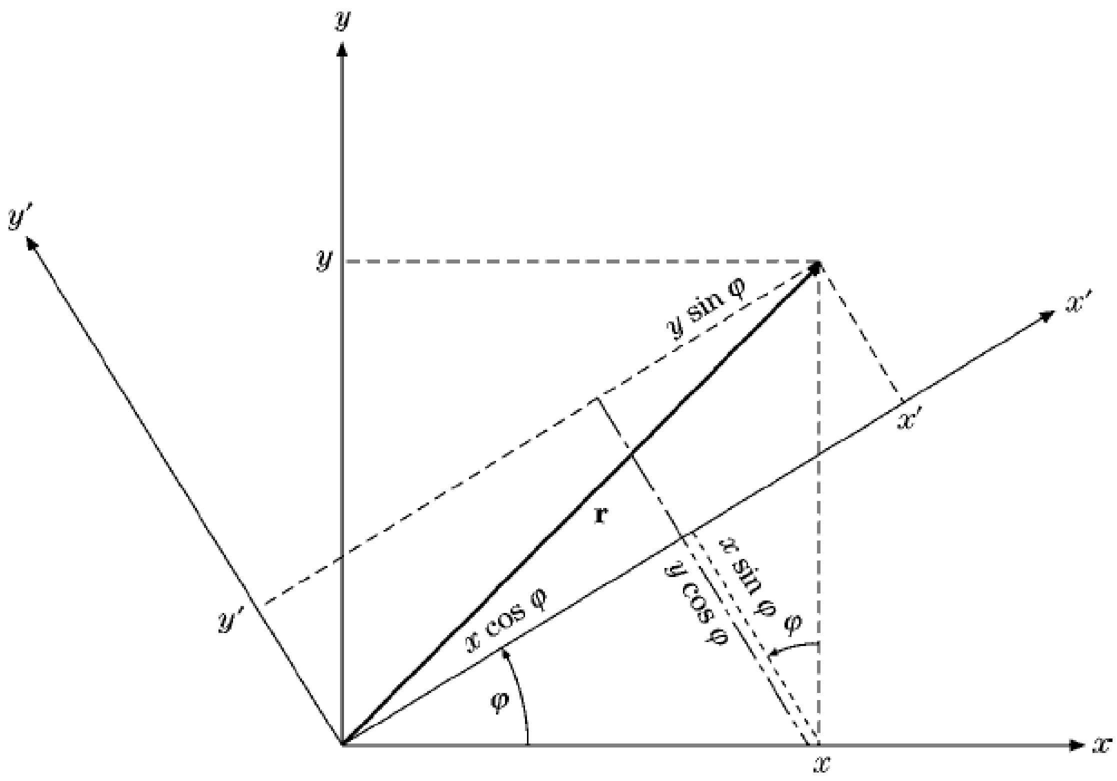

Fig. 6

Transformation of the Components. Now we return to the concept of vector $\vec{r}$ as a geometric object independent of the coordinate system. Let us look at $\vec{r}$ in

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two different systems, one rotated in relation to the other. For simplicity, we consider first the two-dimensional case. If the x, y coordinates are rotated counterclockwise through an angle ϕ, keeping $\vec{r}$, fixed (as shown in Fig. 6), we get the following relations between the components resolved in the original system (unprimed) and those resolved in the new rotated system (primed) as

$$x' = x\cos\phi + y\sin\phi$$

$$y' = -x\sin\phi + y\cos\phi$$

We saw that a vector could be represented by the coordinates of a point, that is, the coordinates were proportional to the vector components. Hence the components of a vector must transform under rotation as coordinates of a point (such as $\vec{r}$). Therefore whenever any pair of quantities A_x and A_y in the x y coordinate system is transformed into $(A'_x,\ A'_y)$ by this rotation of the coordinate system with

$$A'_x = A_x\cos\phi + A_y\sin\phi$$

$$A'_y = -A_x\sin\phi + A_y\cos\phi$$

then we define A_x and A_y as the components of a vector $\vec{A}$. Our vector now is defined in terms of the transformation of its components under rotation of the coordinate system. If A_x and A_y transform in the same way as x and y (the components of the general two-dimensional coordinate vector $\vec{r}$), then they are the components of a vector $\vec{A}$. If A_x and A_y do not show this *form invariance* (also called *covariance*) when the coordinates are rotated, they do not form a vector.

The vector field components A_x and A_y satisfying the defining equations, associate a magnitude A and a direction with each point in space. The magnitude is a scalar quantity, invariant to the rotation of the coordinate system. The direction (relative to the unprimed system) is likewise invariant to the rotation of the coordinate system. The result of all this is that the components of a vector may vary according to the rotation of the primed coordinate system. But the variation with the angle is just such that the components in the rotated

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coordinate system A'_x and A'_y define a vector with the same magnitude and the same direction as the vector defined by the components A_x and A_y relative to the x, y coordinate axes. The components of $\vec{A}$ in a particular coordinate system constitute the representation of $\vec{A}$ in that coordinate system. Thus the transformation relations guarantee that the entity $\vec{A}$ is independent of the rotation of the coordinate system.

This concludes part 1 of this e-report.

The discussion will be continuing in the part 2 of this e-report.

Reference(s):

Mathematical Methods for Physics & Engineering, Riley, Hobson & Bence, Cambridge University Press

Mathematical Methods for Physicists, Arfken & Weber, Elsevier

(All the figures have been collected from the above mentioned references)