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SACT

Introduction :–

Area of a rectangle depends upon its length and breadth, hence we can say that area is the function two variables i.e, its length and breadth.

z is called a function of two variables x and y if z has one definite value for every pair of values of x and y. Symbolically it is written as

$$z = f(x, y)$$

The variable x and y are called independent variables while z is called dependent variables.

Geometrically:

Let, $z = f(x, y)$

where x, y belong to an area A of the xy-plane. For each point (x, y) corresponds a value of z. These values of (x, y, z) from a surface in space.

Hence, the function $z = f(x, y)$ represents a surface.

Limit :–

The function $f(x, y)$ is said to tend to the limit l as a $x \to a$ and $y \to b$ if and only if the limit l is independent of the path ~~flowing~~ by the point (x, y) as $x \to a$ and $y \to b$. Then.

$$\lim_{\substack{x \to a \\ y \to b}} f(x, y) = l.$$

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The function $f(x,y)$ in region R is said to tend to the limit l as $x \to a$ and $y \to b$ if and only if corresponding to a positive number ϵ (a,b) there exists another positive number δ such that

$$|f(x,y) - l| < \epsilon \text{ for } 0 < (x-a)^2 + (y-b)^2 < \delta^2$$

for every point (x,y) in R.

Working Rule To Find The Limit :-

Step 1 : Find the value of $f(x,y)$ along $x \to a$ and $y \to b$.

Step 2 : Find the value of $f(x,y)$ along $y \to b$ and $x \to a$

If the values of $f(x,y)$ in step 1 and step 2 remain the same, the limit exists otherwise not.

Step 3 : If $a \to 0$ and $b \to 0$, find the limit along $y = mx$ or $y = mx^n$. If the values of the limit does not contain m then limit exists. If it contains m, the limit does not exist.

Note : (i) Put $x = 0$ and then $y = 0$ in f. Find its value f_1.

(ii) Put $y = 0$ and then $x = 0$ in f. Find the value f_2.

If $f_1 \neq f_2$, limit does not exist.

If $f_1 = f_2$ then

(iii) Put $y = mx$ and find the limit f_3.

If $f_1 = f_2 \neq f_3$ limit does not exist.

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If $f_1 = f_2 = f_3$, then

(iv) Put $y = mx^2$ and find the limit f_4.

If $f_1 = f_2 = f_3 \neq f_4$ then limit does not exist.

If $f_1 = f_2 = f_3 = f_4$, then limit exists.

● Evalute. $\lim\limits_{\substack{x\to 0\\ y\to a}} \dfrac{x^2y}{x^4+y^2}$.

Sol^{n}:- (i) $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{x^2y}{x^4+y^2} = \lim\limits_{y\to 0} \dfrac{0}{0+y^2} = 0 = f_1$ (say)

(ii) $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{x^2y}{x^4+y^2} = \lim\limits_{x\to 0} \dfrac{0}{x^4+0} = 0 = f_2$ (say)

Hence $f_1 = f_2$, therefore.

(iii) Put $y = mx$

$\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{x^2mx}{x^4+m^2x^2} = \lim\limits_{x\to 0} \dfrac{mx}{x^2+m^2} = 0 = f_3$ (say)

Hence $f_1 = f_2 = f_3$, therefore

(iv) Put $y = mx^2$

$\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{x^2mx^2}{x^4+m^2x^4} = \dfrac{m}{1+m^2} = f_4$

Here $f_1 = f_2 = f_3 \neq f_4$

Thus limit does not exist.

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● Evalute $\lim_{\substack{x\to 0\\ y\to 0}} (x^3+y^3)$.

সমাধান — (i) $\lim_{\substack{x\to 0\\ y\to 0}} (x^3+y^3) = \lim_{y\to 0} (0+y^3) = 0 = f_1$

(ii) $\lim_{\substack{x\to 0\\ y\to 0}} (x^3+y^3) = \lim_{x\to 0} (x^3+0) = 0 = f_2$

Hence $f_1 = f_2$, therefore.

(iii) Put $y = mx$

$$\lim_{\substack{x\to 0\\ y\to 0}} (x^3+y^3) = \lim_{x\to 0} \left[\lim_{y\to mx} (x^3+y^3)\right]$$

$$= \lim_{x\to 0} (x^3+m^3x^3) = 0 = f_3 \text{ (say)}$$

Hence $f_1 = f_2 = f_3$, therefore

(iv) Put $y = mx^2$

$$\lim_{\substack{x\to 0\\ y\to 0}} (x^3+y^3) = \lim_{x\to 0} \left[\lim_{y\to mx^2} (x^3+y^3)\right]$$

$$= \lim_{x\to 0} (x^3+m^3x^6)$$

$$= \lim_{x\to 0} x^3(1+m^3x^3)$$

$$= 0 = f_4$$

Here $f_1 = f_2 = f_3 = f_4$

Thus, limit exists with value 0.

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● **Evaluate :** $\lim\limits_{\substack{x\to 1\\ y\to 2}} \dfrac{3x^2y}{x^2+y^2+5}$.

Sol^n :– $\lim\limits_{\substack{x\to 1\\ y\to 2}} \dfrac{3x^2y}{x^2+y^2+5} = \lim\limits_{x\to 1}\left[\lim\limits_{y\to 2} \dfrac{3x^2y}{x^2+y^2+5}\right] = \lim\limits_{x\to 1} \dfrac{3x^2(2)}{x^2+(2)^2+5}$

$= \lim\limits_{x\to 1} \dfrac{6x^2}{x^2+9} = \dfrac{6}{1+9} = \dfrac{3}{5}$

● **Evaluate :** $\lim\limits_{\substack{x\to \infty\\ y\to 3}} \dfrac{2x-3}{x^3+4y^3}$.

Sol^n :– (i) $\lim\limits_{\substack{x\to \infty\\ y\to 3}} \dfrac{2x-3}{x^3+4y^3} = \lim\limits_{y\to 3}\left[\lim\limits_{x\to \infty} \dfrac{2x-3}{x^3+4y^3}\right]$

$= \lim\limits_{y\to 3}\left[\lim\limits_{x\to \infty} \dfrac{2/x^2 - 3/x^3}{1+4(y/x)^3}\right]$

$= \lim\limits_{y\to 3}\left(\dfrac{0-0}{1+4(0)}\right) = 0 = f_1$

(ii) $\lim\limits_{\substack{y\to 3\\ x\to \infty}} \dfrac{2x-3}{x^3+4y^3} = \lim\limits_{x\to \infty}\left[\lim\limits_{y\to 3} \dfrac{2x-3}{x^3+4y^3}\right]$

$= \lim\limits_{x\to \infty} \dfrac{2x-3}{x^3+108}$

$= \lim\limits_{x\to \infty} \dfrac{2/x^2 - 3/x^3}{1+108/x^3} = \dfrac{0-0}{1+0} = 0 = f_2$

Here $f_1 = f_2$

Hence, the limit exists with value 0.

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Continuity :—

A function $f(x,y)$ is said to be continuous at a point (a,b) if

$$\lim_{\substack{x\to a\\ y\to b}} f(x,y) = f(a,b)$$

A function is said to be continuous in a domain at every point of the domain.

Working Rule for Continuity at a Point (a,b) :—

Step 1 :- $f(a,b)$ should be well defined

Step 2 :- $\lim_{(x,y)\to(a,b)} f(x,y)$ should exist.

Step 3 :- $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Test the function $f(x,y) = \begin{cases} \frac{x^3-y^3}{x^2+y^2} & \text{when } x\neq 0, y\neq 0 \\ 0 & \text{when } x=0, y=0 \end{cases}$

Sol^{n} :-

Step 1 :- The function is well defined at $(0,0)$.

Step 2 :- $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = \lim_{x\to 0}\left[\lim_{y=mx} \frac{x^3-y^3}{x^2+y^2}\right]$

$$= \lim_{x\to 0} \frac{x^3 - m^3x^3}{x^2+m^2x^2} = \lim_{x\to 0} \frac{x(1-m^3)}{1+m^2} = 0$$

Thus limit exists at $(0,0)$.

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Step 3 :- limit of $f(x)$ at origin = value of the function at origin

$$\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = f(0,0) = 0$$

Hence the function f is continuous at the origin.

② Discuss the continuity of $f(x,y) = \begin{cases} \frac{x}{\sqrt{x^2+y^2}} & , x\neq 0, y\neq 0 \\ 2 & x=0, y=0 \end{cases}$

Sol^n :-

Step 1 :- The function $f(x,y)$ at $(0,0)$ is well defined.

Step 2 : $$\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}} = \lim_{x\to 0}\left[\lim_{y\to mx} \frac{x}{\sqrt{x^2+y^2}}\right]$$

$$= \lim_{x\to 0} \frac{x}{\sqrt{x^2+m^2x^2}}$$

$$= \lim_{x\to 0} \frac{1}{\sqrt{1+m^2}}$$

for different values of m the limit f is not unique

So the, $\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}}$ does not exist.

Hence $f(x,y)$ is not continuous at origin.

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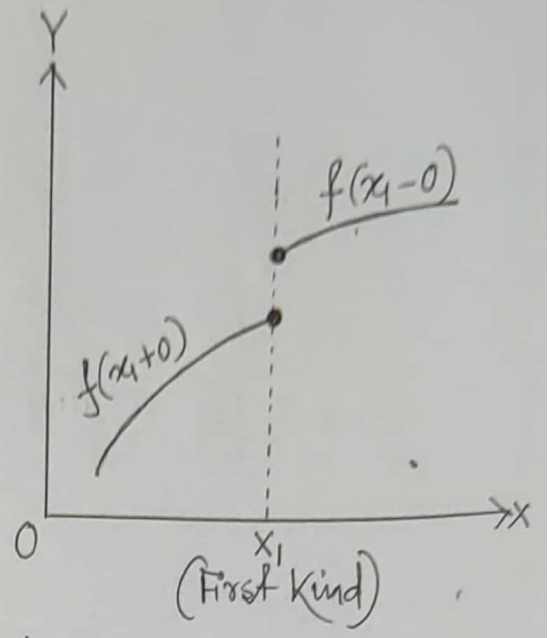
Types of Discontinuity :—

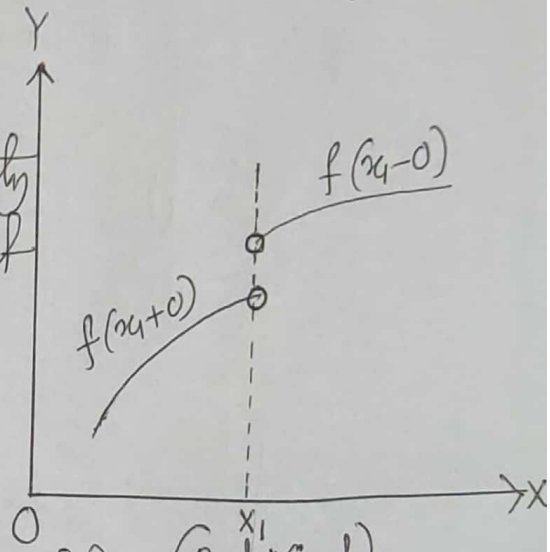
1. First Kind :—

$f(x)$ is said to have discontinuity of first kind at the point $x = x_1$ if Right limit (x_1+0) and left limit (x_1-0) exist but are not equal.



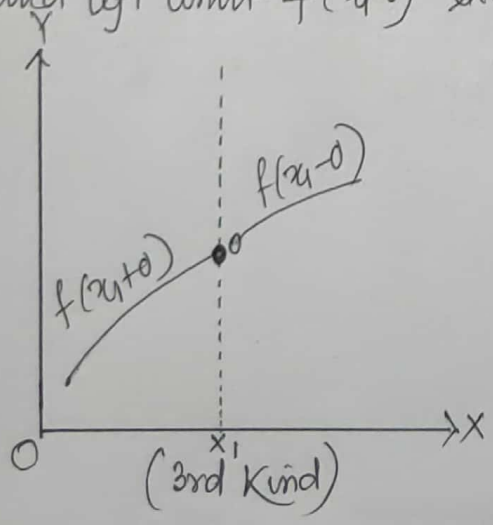
2. Second Kind :—

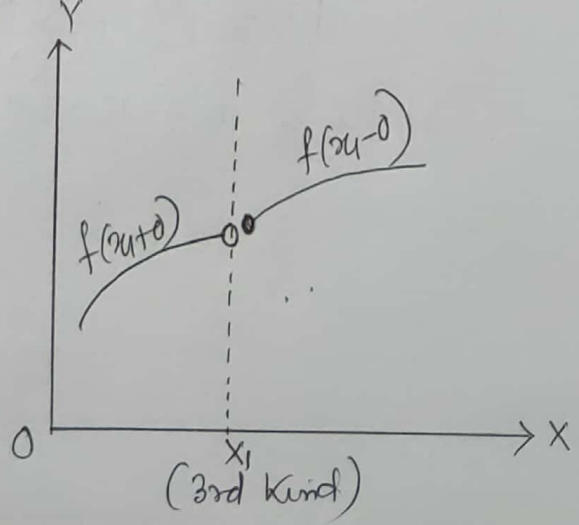
$f(x)$ is said to have discontinuity at $x = x_1$ if neither right limit $f(x_1+0)$ exists nor left limit $f(x_1-0)$ exists.



3. Third Kind (Mixed discontinuity) :—

$f(x)$ is said to have mixed discontinuity at the point $x = x_1$ if only one of the two limits right limit $f(x_1+0)$ and left limit $f(x_1-0)$ exists and not the other.





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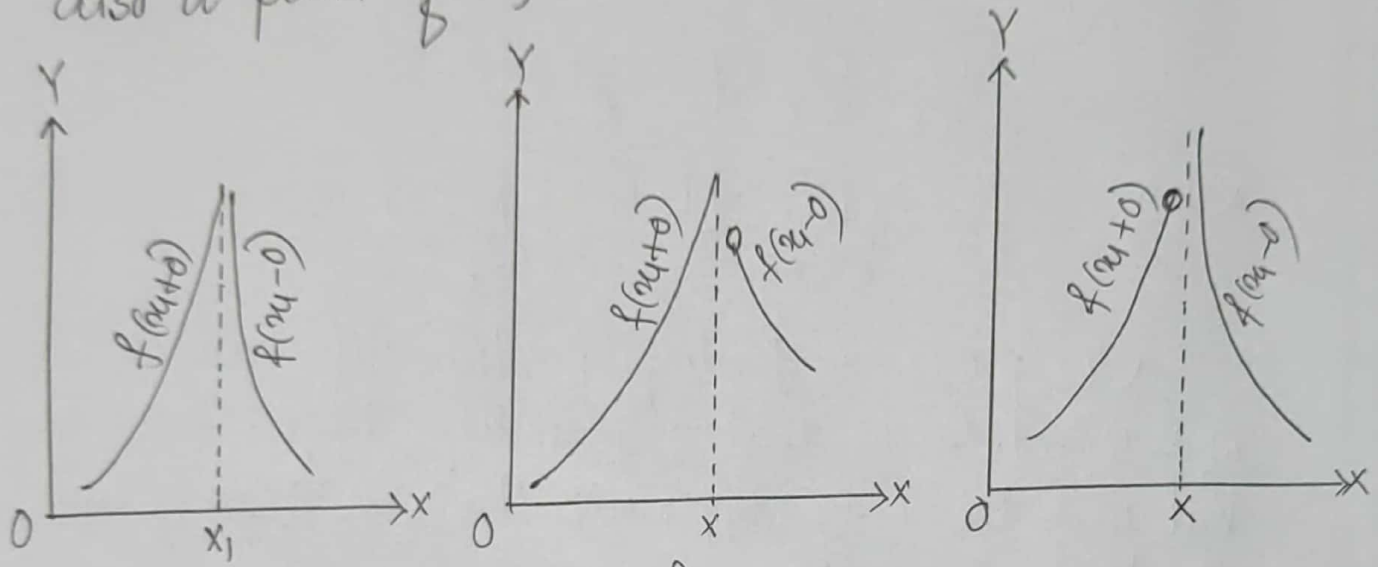
4. Fourth Kind (Infinite discontinuity) :—

$f(x)$ is said to have infinite discontinuity at the point $x = x_1$, if either one or both limits right limit and left limit $f(x_1 - 0)$ is infinite.

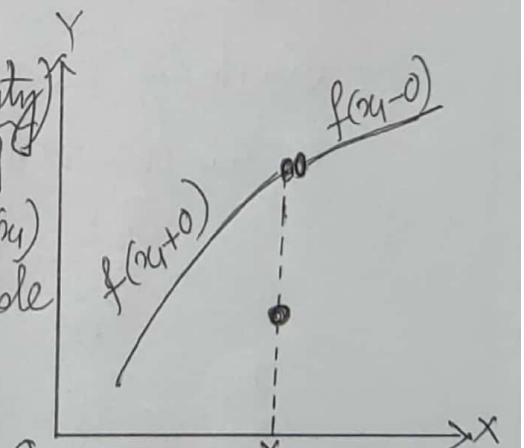
If both limits do not exists and if $f(x_1 \pm h)$ oscillates between limits one of which is infinite as $\pm h \to 0$. It is also a point of infinite discontinuity.



(Fourth Kind)

5. Fifth Kind :— (Removable discontinuity)

If right limit $f(x_1+0)$ is equal to left limit $f(x_1-0)$ is not equal to $f(x_1)$ then $f(x)$ is said to have removable discontinuity.



(Fifth Kind)

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