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Paper: DSE1T (Classical Dynamics)

Topic: Classical Mechanics of Point Particles (Part-2)

Conservation Laws: In continuation to the previous study material, we are going to discuss here the conservation laws for a particle in motion. Conservation laws basically tell us the conditions under which the dynamical quantities like linear momentum, angular momentum, energy etc. are conserved. We will use Newton's laws (stated in the previous material) to deduce the conservation laws.

Conservation of Linear Momentum:

Let us consider that a particle of mass m is subjected to a force $\vec{F}$. According to Newton's 2nd law of motion:

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) \qquad \text{—— (6)}$$

where $\vec{p} = m\vec{v}$ is the linear momentum of the particle. If we now consider the external force, acting on the particle, to be zero, then

$$\frac{d\vec{p}}{dt} = 0$$

$$\text{or, } \vec{p} = m\vec{v} = \text{constant}$$

Therefore, in the absence of an external force, the linear momentum of a particle ~~is~~ remains constant in time. This is the conservation theorem for a free particle.

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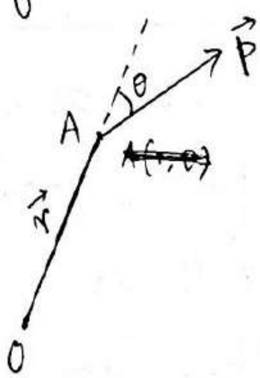
Conservation of Angular Momentum :

Angular momentum of a particle of mass m about a point O (as shown in the figure) is defined as



$$\vec{J} = \vec{r} \times \vec{p} \quad \text{—— (8)}$$

where $\vec{r}$ is the position vector of the particle and $\vec{p} = m\vec{v}$ is the ~~m~~ linear momentum of it. ~~Ef~~ The torque (about O) acting on the particle is defined as

$$\vec{\tau} = \vec{r} \times \vec{F} \quad \text{—— (9)}$$

where $\vec{F}$ is the force acting on the particle.

Now, let us calculate ~~take~~ the derivative of ~~eqn~~ the angular momentum with respect to time.

$$\frac{d\vec{J}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) \qquad [\text{Using eqn. (8)}]$$

$$= \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

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$$\text{or, } \frac{d\vec{J}}{dt} = \vec{v} \times (m\vec{v}) + \vec{r} \times \vec{F} \qquad [\text{As } \vec{p} = m\vec{v}]$$

$$= \vec{r} \times \vec{F} \qquad [\text{As } \vec{v} \times \vec{v} = 0]$$

$$= \vec{\tau} \qquad \text{——— (10)}$$

Now, we can define the torque as the time rate of change of angular momentum. So, torque is analogous to force i.e. torque does the same job in angular motion what force does in linear motion.

In short: Force $(\vec{F})$ = Rate of change of linear momentum $\left(\frac{d\vec{p}}{dt}\right)$

Torque $(\vec{\tau})$ = Rate of change of angular momentum $\left(\frac{d\vec{J}}{dt}\right)$

If there is no torque acting on the particle, then

$$\vec{\tau} = \frac{d\vec{J}}{dt} = 0$$

$$\text{or, } \vec{J} = \text{constant}$$

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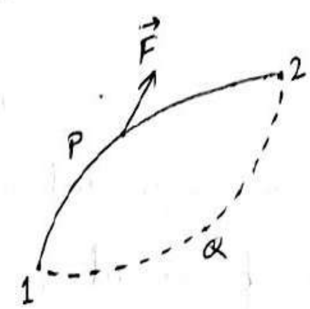
Therefore, in the absence of an external torque, the angular momentum of a particle is a constant of motion. This is the conservation theorem of angular momentum of a particle.

Conservation of Energy :

If a particle is moved from point 1 ~~as~~ to point 2 by an external force $\vec{F}$ then the work done by the force is defined as



$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{r} \quad \text{——} (11)$$

Now, according to Newton's 2nd law of motion, we have $\vec{F} = m \frac{d\vec{v}}{dt}$. So,

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{r} = \int_1^2 m \frac{d\vec{v}}{dt} \cdot d\vec{r}$$

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$$= \int_1^2 m \frac{d\vec{v}}{dt} \cdot \vec{v}\, dt \quad \left[\because \vec{v} = \frac{d\vec{r}}{dt}\right]$$

$$= \int_1^2 \frac{1}{2} m \frac{d}{dt}(\vec{v} \cdot \vec{v})\, dt$$

$$= \int_1^2 d\left(\frac{1}{2} m v^2\right)$$

$$= \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2$$

The scalar quantity $\frac{1}{2}mv^2$ is termed as kinetic energy and it is denoted by T.

$$\therefore W_{12} = T_2 - T_1 \qquad \text{——— (12)}$$

In words: Workdone by the force acting on the particle is equal to the change in the kinetic energy. This is known as work-energy theorem.

Now, we will define conservative force and then potential energy.

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Conservative force: If a force does the same amount of work in moving a particle from a point to another particular point through all possible paths between the points, then the force is called the conservative force. Therefore, if the force in the fig. on the last page is conservative then

$$\int_{1 \atop \text{path P}}^{2} \vec{F}\cdot d\vec{r} = \int_{1 \atop \text{path Q}}^{2} \vec{F}\cdot d\vec{r}$$

$$\text{or,} \quad \int_{1 \atop \text{path}}^{2} \vec{F}\cdot d\vec{r} + \int_{2 \atop \text{path Q}}^{1} \vec{F}\cdot d\vec{r} = 0$$

$$\text{or,} \quad \oint \vec{F}\cdot d\vec{r} = 0 \qquad \text{——— (13)}$$

So, for a conservative force, the workdone on the particle around a closed path is zero.

Let us now apply Stoke's theorem on the L.H.S. of eqn. (13),

$$\iint (\vec{\nabla}\times\vec{F})\cdot d\vec{s} = 0 \qquad \text{——— (14)}$$

[Integration is over the surface enclosed by the closed path]

Now, as $d\vec{s}$ is elementary surface area, so it can't be zero.

$$\therefore \vec{\nabla} \times \vec{F} = 0 \qquad (15)$$

From the vector analysis we know that if curl of any vector is zero then that vector can be expressed as the gradient of a scalar function i.e.

If say, $\vec{\nabla} \times \vec{A} = 0$ then $\vec{A} = \pm \vec{\nabla}\phi$

as $\vec{\nabla} \times (\pm \vec{\nabla}\phi) = 0$ always.

So, for a conservative force we write

$$\vec{F} = -\vec{\nabla} V \qquad (16)$$

$$\text{or, } \vec{F} = -\left(\hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right)$$

The negative sign $(-)$ is choosen for convenience. The scalar function V is called the potential energy and it depends on the position. Now, for conservative force,

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{r} = -\int_1^2 \vec{\nabla} V \cdot d\vec{r}$$

$$= -\int_1^2 dV$$

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or, $$W_{12} = V_1 - V_2 \qquad (17)$$

So, the work done is equal to the change in the potential energy of the particle. Now, eqn. (12) and eqn. (17) gives us

$$T_2 - T_1 = V_1 - V_2$$

or, $$T_1 + V_1 = T_2 + V_2 \qquad (18)$$

So, the sum of the kinetic energy and the potential energy of a particle moving under a conservative force remains constant. The sum of the kinetic energy and the potential energy is called the total mechanical energy (E). Therefore, in a conservative force field

$$T + V = E = \text{constant}.$$

This is known as ~~cons~~ the law of conservation of energy. Total energy E is a constant of motion (or also called first integral of the motion) in a conservative force field.

Reference:

1. *Classical Mechanics - J. C. Upadhyaya*
2. *Classical Mechanics - N. C. Rana & P. S. Joag*