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DSE2T (Nuclear and Particle Physics)

Topic – Particle Physics (Part – 3)

We have already discussed part 2 of this e-report.

Now let us continue part 3 of it.

Symmetry and Conservation Law:

As we saw in the previous e-report, although several quantum numbers appear to be conserved in strong high-energy processes, some are violated in weak and electromagnetic interactions. This must reflect the inherent character of the underlying forces. Consequently, understanding the origin of conservation principles, and under what conditions they are violated, would appear to be an important element in the formulation of a quantitative description of particle interactions. We will therefore first address the question of how conservation laws arise in physical theories.

Noether's Theorem. The simple answer to this query comes from *Noether's Theorem*, and the answer is that whenever there is an underlying symmetry in a physical system, that means if the system is not affected by a change in some coordinate or other dynamical variable, then we can define a conserved quantum number associated with that symmetry. Conversely, if there is a conserved quantity associated with a physical system, then there exists an underlying invariance or symmetry principle responsible for its conservation. This observation, known as Noether's Theorem gives rise to powerful restrictions on the structure of physical theories.

Symmetries in the Lagrangian Formalism:

In simple terms, any set of transformations that leaves the equations of motion of a system unchanged or invariant, defines what is known as a symmetry of that physical system. Symmetries can be discussed using either the Lagrangian or the Hamiltonian formalism, for both classical as well as quantum theories.

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We choose the discussion using the Lagrangian framework, which is most appropriate for studying relativistic systems.

The Lagrangian (L) for a general system with n-degrees of freedom (e.g., n-coordinates and n-velocities) can be represented as $L = L(q_i, p_i)$, for $i = 1$, 2,…,n. The momenta (p_i) associated with or conjugate to, the coordinates q_i can be defined as given $p_i = \frac{\partial L}{\partial \dot{q}_i}$, for $i = 1, 2, \ldots, n$ and the general dynamical equations of motion can be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

$$\text{or } \frac{dp_i}{dt} - \frac{\partial L}{\partial q_i} = 0$$

Now Let us suppose that the Lagrangian for a given physical system is independent of some particular coordinate q_m. It then follows that $\frac{\partial L}{\partial q_m} = 0$ for a specific value of m. As a result, the dynamical equation will give us

$$\frac{dp_m}{dt} = 0$$

$$\text{or } p_m = \text{constant}$$

In other words, if the Lagrangian for a physical system does not depend explicitly on a given coordinate, then the corresponding conjugate momentum is conserved. Moreover, if a Lagrangian does not depend on some particular coordinate, it must be invariant under translations (redefinitions) of this coordinate, which brings to focus the connection between the invariance of a theory and a corresponding conserved quantity.

As a simple demonstration of these ideas, let us consider the motion of a free rotor. In the absence of any force, the system has only kinetic energy and we can therefore write

$$L = T = \frac{1}{2} I \dot{\theta}^2$$

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where I denotes the moment of inertia of the rotor and $\dot{\theta}$ its angular velocity. This Lagrangian is independent of the angular coordinate θ of the rotor, and correspondingly we conclude that

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta} = \text{constant}$$

Hence, the lack of an explicit θ-dependence in the Lagrangian of a rotor gives rise to the rotational invariance of the system, and leads to a constant value of its angular momentum (p_θ).

Such conclusions are quite general and we summarize in Table 1 several common transformations and the associated quantities that are conserved when physical systems are invariant under these transformations. The converse argument also holds, namely that for every conserved quantity of a physical system, there exists an underlying invariance principle.

Type of Transformation	Conserved Quantity
Space Translation	Momentum
Time Translation	Energy
Rotation in real space	Angular Momentum
Gauge Transformation	Charge
Rotation in isospin space	Isospin

Table 1

Symmetries in Quantum Mechanics:

The transition from classical mechanics to quantum mechanics is best described within the framework of the Hamiltonian formalism. In quantum mechanics, classical observables are represented by Hermitian operators, and the Poisson brackets are replaced by appropriate commutation relations. The classical generators of infinitesimal transformations therefore become operators that define symmetry transformations for operators as well as for vectors in Hilbert space. In quantum theory, such symmetry transformations can be implemented in one of the two equivalent ways, namely, either by transforming the state vectors in the Hilbert space or by transforming the operators that act on them.

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This is quite similar to the two ways that a classical transformation can be implemented, namely, as a passive or an active transformation.

In quantum mechanical language, any observable quantity corresponds to the expectation value of a Hermitian operator in a given quantum state, and its time evolution is given by Ehrenfest's Theorem, given by

$$\frac{d}{dt}\langle A\rangle = \frac{1}{i\hbar}\langle[A,H]\rangle$$

where we have denoted the expectation value of an operator A in a state $|\psi\rangle$ as

$$\langle A\rangle = \langle\psi|A|\psi\rangle$$

It is understood that an observable quantity that does not depend explicitly on time will be conserved if and only if the corresponding quantum operator commutes with the Hamiltonian (H). That is, for any quantum state, we will obtain $\frac{d}{dt}\langle A\rangle = 0$, if and only if $[A,H] = 0$.

We conclude that the infinitesimal transformations generated by an operator A define a symmetry of the theory when the previous commutation relation holds. As a consequence of the symmetry, the expectation value of A in any quantum state is independent of time (i.e. is conserved). Conversely, when an observable or the expectation value of A in any quantum state is conserved (is constant in time), then A generates a symmetry of the underlying physical system. In quantum mechanics, when two operators commute, they can be diagonalized simultaneously, that is, they can have a complete set of common eigen-vectors. Thus, when the Hamiltonian has an underlying symmetry defined by the generator A, the energy eigen-states are also eigen-vectors of the operator A, and can also be labelled by the quantum numbers corresponding to the eigenvalues of A. Moreover, these quantum numbers are conserved in any physical process where the interaction Hamiltonian for some transition (e.g., decay or reaction) is invariant under the symmetry transformation. However, for transitions in which interaction Hamiltonians are not invariant under symmetry transformations, the corresponding quantum numbers do not have to be conserved. This provides an understanding of why some quantum numbers are conserved whereas others are violated in different interactions, and points to an essential first step in constructing physical theories of fundamental interactions.

Continuous Symmetries:

Broadly speaking, all symmetry transformations of a theory can be classified into two categories: those that depend on a continuous set of parameters and those that correspond to some discrete jumps. Accordingly, they are known, respectively, as *continuous* and *discrete transformations*. All the examples of symmetry transformations that we have considered thus far in this chapter can be identified with continuous transformations, since they depend on an arbitrary parameter of the transformation (e.g. ϵ). We will now proceed further with our development of *continuous symmetries*.

For continuous transformations, the infinitesimal transformation has fundamental importance because any finite transformation can be described in terms of a series of successive infinitesimal transformations. This can be shown as follows. Let us consider an infinitesimal translation of the x-coordinate by a constant amount ϵ. Thus, for $x \to x' = x + \epsilon$, with ϵ real, our wave function corresponding to a given state vector changes as

$$\psi(x) \to \psi(x-\epsilon) \approx \psi(x) - \epsilon \frac{d\psi(x)}{dx}$$

Consequently, it can be shown that under this transformation, the expectation value of the Hamiltonian changes as follows

$$\langle H \rangle \to \langle H \rangle' \approx \langle H \rangle - \epsilon \int_{-\infty}^{\infty} \psi^*(x)\left(H\frac{d}{dx} - \frac{d}{dx}H\right)\psi(x)dx$$

$$\text{or } \langle H \rangle' \approx \langle H \rangle - \frac{i\epsilon}{\hbar}\int_{-\infty}^{\infty} \psi^*(x)(Hp_x - p_x H)\psi(x)dx$$

$$\text{or } \langle H \rangle' \approx \langle H \rangle - \frac{i\epsilon}{\hbar}\langle [H, p_x] \rangle$$

where we have used the momentum operator p_x defined as $p_x = \frac{\hbar}{i}\frac{d}{dx}$.

This result shows that, to first order in ϵ, the quantum generator of infinitesimal space translations G can be identified with the momentum operator, namely,

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$$\epsilon G = -\frac{i\epsilon}{\hbar} p_x$$

Also the Hamiltonian will be invariant under translations of the x coordinate if

$$[p_x, H] = 0$$

We notice that the effect of an infinitesimal translation ϵ along the x-axis on a state $|\psi\rangle$ is given by the operator $U_x(\epsilon)$, where

$$\psi(x-\epsilon) = U_x(\epsilon)\psi(x) \approx \psi(x) - \epsilon \frac{d\psi(x)}{dx}$$

$$\text{or } U_x(\epsilon)\psi(x) = (1 - \frac{i\epsilon}{\hbar} p_x)\psi(x)$$

$$\text{or } U_x(\epsilon) = 1 - \frac{i\epsilon}{\hbar} p_x$$

The operator corresponding to a finite translation along the x-axis, namely $U_x(\alpha)$ (where α is no longer infinitesimal), can be obtained as follows. Let us consider N successive infinitesimal translations by an amount ϵ along the x-axis. This corresponds to a total translation by an amount $N\epsilon$, and the operator representing such a transformation corresponds merely to the product of N infinitesimal translations applied in succession.

So, we can write

$$U_x(N\epsilon) = \left(1 - \frac{i\epsilon}{\hbar} p_x\right)\left(1 - \frac{i\epsilon}{\hbar} p_x\right) \ldots \left(1 - \frac{i\epsilon}{\hbar} p_x\right) = \left(1 - \frac{i\epsilon}{\hbar} p_x\right)^N$$

Therefore,

$$U_x(\alpha) = \lim_{N\to\infty} \left(1 - \frac{i\epsilon}{\hbar} p_x\right)^N = \lim_{N\to\infty} \left(1 - \frac{i\alpha}{N\hbar} p_x\right)^N = e^{-\frac{i\alpha p_x}{\hbar}} = e^{\alpha G}$$

We finally see that the operator for finite transformations is obtained simply by exponentiating the generators of infinitesimal transformations.

Isospin Symmetry. we will now expand on the application of the previous ideas to isospin. If there is an isospin symmetry, then the implication is that our "isospin up" proton (p) with $I_3 = \frac{1}{2}$ and our "isospin down" neutron (n) with

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$I_3 = -\frac{1}{2}$ are indistinguishable. We denote these states as the $|p\rangle$ and $|n\rangle$. We can consequently define a new neutron and proton state as some linear superposition of the $|p\rangle$ and $|n\rangle$ vectors. We see that a finite rotation of our vectors in an *isospin space* by an arbitrary angle θ about the I_2 axis leads to a set of transformed vectors $|p'\rangle$ and $|n'\rangle$ where

$$|p'\rangle = \cos(\frac{\theta}{2})\,|p\rangle - \sin(\frac{\theta}{2})|n\rangle$$

$$|n'\rangle = \sin(\frac{\theta}{2})|p\rangle + \cos(\frac{\theta}{2})\,|n\rangle$$

Now, let us see what such an invariance implies about the nucleon-nucleon interaction. Our two-nucleon quantum states in the Hilbert space can be written in terms of the more fundamental states, which are either symmetric or antisymmetric under an exchange of particles. These correspond to the following four states

$$|\psi_1\rangle = |pp\rangle,\ |\psi_2\rangle = \frac{1}{\sqrt{2}}(|pn\rangle + |np\rangle)$$

$$|\psi_3\rangle = |nn\rangle,\ |\psi_4\rangle = \frac{1}{\sqrt{2}}(|pn\rangle - |np\rangle)$$

Assuming that I_3 is an additive quantum number, we can identify the isospin projections as: for $|\psi_1\rangle$ $I_3 = +1$, for $|\psi_2\rangle$ and $|\psi_4\rangle$ $I_3 = 0$ and for $|\psi_3\rangle$ $I_3 = -1$. Let us look into the states $|\psi_1\rangle$ and $|\psi_4\rangle$ under the rotation in isospin space.

$$|\psi_1{}'\rangle = |\{\cos(\frac{\theta}{2})\,p - \sin(\frac{\theta}{2})n\}\{\cos(\frac{\theta}{2})\,p - \sin(\frac{\theta}{2})n\}\rangle$$

$$= \cos^2(\frac{\theta}{2})|pp\rangle - \cos(\frac{\theta}{2})\sin(\frac{\theta}{2})(|pn\rangle + |np\rangle) + \sin^2(\frac{\theta}{2})|nn\rangle$$

$$= \cos^2(\frac{\theta}{2})|\psi_1\rangle - \sqrt{2}\cos\left(\frac{\theta}{2}\right)\sin(\frac{\theta}{2})|\psi_2\rangle + \sin^2(\frac{\theta}{2})|\psi_3\rangle$$

and

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$$|\psi_4'\rangle = \frac{1}{\sqrt{2}}|\{\cos(\frac{\theta}{2})\,p - \sin(\frac{\theta}{2})n\}\{\sin(\frac{\theta}{2})p + \cos(\frac{\theta}{2})\,n\}\rangle$$

$$-\frac{1}{\sqrt{2}}|\{\sin(\frac{\theta}{2})p + \cos(\frac{\theta}{2})\,n\}\{\cos(\frac{\theta}{2})\,p - \sin(\frac{\theta}{2})n\}\rangle$$

$$= \frac{1}{\sqrt{2}}(\cos^2(\frac{\theta}{2}) + \sin^2(\frac{\theta}{2}))(|pn\rangle - |np\rangle) = |\psi_4\rangle$$

Therefore we obtain $|\psi_4'\rangle = |\psi_4\rangle$. We see that $|\psi_4\rangle$ is totally insensitive to rotations in this space. It must consequently correspond to a "singlet" combination, and represent the $I = 0$, $I_3 = 0$ nucleon-nucleon system. We can also calculate the changes in the states $|\psi_2\rangle$ and $|\psi_3\rangle$ under the above rotation, and show that three remaining states transform into one another under the isospin rotation, just as the three components of a vector do under a spatial rotation. If there is any isospin invariance in the nucleon-nucleon strong interaction, then it follows that the three states $|\psi_1\rangle$, $|\psi_2\rangle$ and $|\psi_3\rangle$, corresponding to $I_3 = 1$, 0, and -1, respectively, are equivalent and cannot be distinguished from each other. Consequently, it appears that any two nucleon system can be classified either as an $I = 0$ singlet or an $I = 1$ triplet in isotopic spin space. The singlet and the three triplet states are independent of each other, and the three substates of $I = 1$ are indistinguishable if isospin is a symmetry of the system, that is, if the nucleon-nucleon strong interaction is not sensitive to the replacement of a neutron by a proton. Any breaking of the degeneracy of the $I = 1$ states must arise from other interactions.

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This concludes part 3 of this e-report.

The discussion will be continuing in the part 4 of this e-report.

Reference(s):

Introduction to Elementary Particles, D. J. Griffiths, John Wiley & Sons

Introduction to Nuclear and Particle Physics, A. Das & T. Ferbel, World Scientific

(All the figures have been collected from the above mentioned references)