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Paper: C5T (Mathematical Physics II)
Topic: Fourier Series

Fourier series is a mathematical tool for solving differential equations with periodic boundary conditions. It has many applications in many branches of Physics, for example, optics, acoustics, electronics, electrodynamics etc. So, we are going to learn it in great detail. But, before introducing the Fourier series, we will briefly discuss the Periodic Functions and some of their properties.

Periodic Functions: A function $f(x)$ is said to be periodic if for all x the following condition is satisfied:

$$f(x+T) = f(x) \quad \text{——— (1)}$$

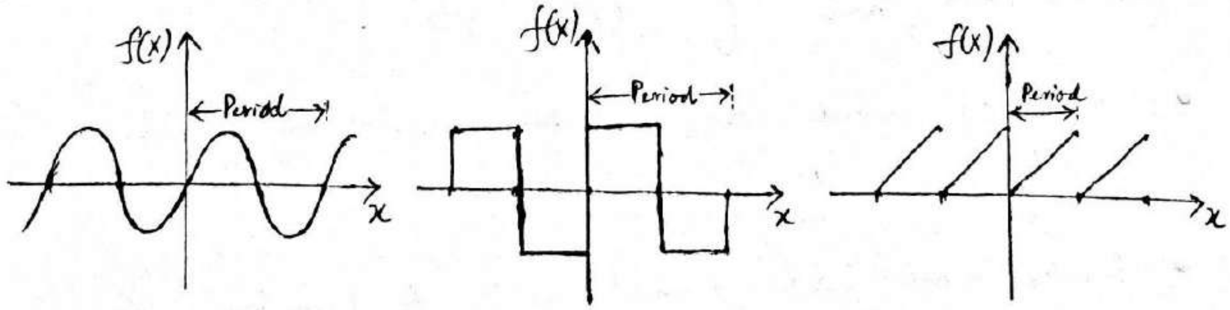
where T is a positive constant and called the period of the function $f(x)$. The minimum value of $T(>0)$ is called the least period or it is simply called the period of the function.

Examples: (i) $f(x) = \sin x$ has periods $2\pi, 4\pi, 6\pi, \ldots$ as $\sin(x+2\pi) = \sin(x+4\pi) = \cdots = \sin x$. However, 2π is the least period or the period of $\sin x$.

(ii) $f(x) = \cos nx$, n = positive integer has period of $2\pi/n$.

(iii) $f(x) = \tan x$ has the period of π.

Some pertdic functions are shown in the figures below:



There are some interesting facts about periodic functions; some of them are discussed below:

- If $f(x)$ and $g(x)$ are periodic functions with period T then $(f+g)$ and fg ~~both~~ ~~will~~ are also periodic with the same period T.

Proof: Let's say, $f(x) + g(x) = F(x)$

Now, $F(x+T) = f(x+T) + g(x+T)$

$= f(x) + g(x)$ [As $f(x)$ and $g(x)$ are periodic with period T]

$= F(x)$

So, $F(x)$ i.e $(f+g)$ is periodic with period T.

Now, assume $f(x)\,g(x) = M(x)$.

$$\therefore M(x+T) = f(x+T)\,g(x+T)$$
$$= f(x)\,g(x)$$
$$= M(x)$$

Hence, $f(x)\,g(x)$ is also periodic with the period same as that of $f(x)$ or $g(x)$.

- $\sin(\omega x)$ and $\cos(\omega x)$ are periodic functions with period $T = \frac{2\pi}{\omega}$.

- <u>Even and Odd functions</u>: A function $f(x)$ is said to be even if

~~Even~~

$$f(-x) = f(x).$$

And a function $\cancel{f}\, g(x)$ is said to be odd if

$$g(-x) = -g(x).$$

<u>Examples</u>: Even functions: $f_1(x) = x^2$, $f_2(x) = \cos x$ etc.

Odd functions: $g_1(x) = x$, $g_2(x) = \sin x$ etc.

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Important properties of even and odd functions are :

For even function : $\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx$

For odd function : $\int_{-b}^{b} g(x)\, dx = 0$

These are very easy to prove and these are valid for symmetric interval like $[-a, a]$ or $[-b, b]$.

Orthogonal functions :

(i) Two functions $f(x)$ and $g(x)$ are said to be orthogonal on $a \leq x \leq b$ if

$$\int_{a}^{b} f(x)\, g(x)\, dx = 0 \qquad \text{--- (2)}$$

(ii) A set of non-zero functions $\{f_i(x)\}$ is said to be mutually orthogonal on $a \leq x \leq b$ if $f_i(x)$ and $f_j(x)$ are orthogonal for every $i \neq j$ or mathematically,

$$\int_{a}^{b} f_i(x)\, f_j(x)\, dx = \begin{cases} 0 & \text{for } i \neq j \\ c>0 & \text{for } i = j \end{cases} \qquad \text{--- (3)}$$

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Orthogonality of sine and cosine functions:

Sine and cosine functions have the following orthogonality properties:

(a) $$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \end{cases} \quad \text{---} (4)$$

(b) $$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \end{cases} \quad \text{---} (5)$$

(c) $$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = 0 . \quad \text{---} (6)$$

Here, m and n are positive integers.

I want you, students, to calculate the above integrations and verify the results.

Fourier Series: A Fourier series is defined as an expansion or representation of a function in an infinite series of sines and cosines. If $f(x)$ be defined in the interval $(-L, L)$ and outside the interval by $f(x+2L) = f(x)$ i.e $f(x)$ is periodic function with the period $2L$. Then, the Fourier series of $f(x)$ can be written as

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$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \qquad \text{—}(7)$$

where a_n and b_n are called the Fourier coefficients and are given by

$$\left.\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \end{aligned}\right\} n = 0, 1, 2, \ldots \qquad \text{—}(8)$$

Notice that a_0 is singled out and a $\frac{1}{2}$ factor is ~~mu~~ included. It is done to ~~be~~ make eqn. (8) valid for all n i.e for $n=0$ and $n>0$. Also, note that ~~bn~~ $b_0 = 0$ always.

<u>Dirichlet Conditions</u>: Dirichlet conditions are sufficient conditions for a real-valued periodic function $f(x)$ to be equal to the sum of its Fourier series at each point where $f(x)$ is continuous. And the sum of the Fourier series at the points of discontinuity will be equal to the average of the values of the discontinuity. The conditions are following:

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(i) $f(x)$ is defined except possibly at a finite number of points in $(-L, L)$

(ii) $f(x)$ is periodic outside $(-L, L)$ with period $2L$.

(iii) $f(x)$ and $f'(x)$ are piecewise continuous in $(-L, L)$.

So, if $f(x)$ satisfies the above conditions then the Fourier series (eqn. 7) converges to

(a) $f(x)$ if x is a point of continuity

(b) $\dfrac{f_+(x) + f_-(x)}{2}$ if x is a point of discontinuity.

$f_+(x)$ and $f_-(x)$ represent the right- and left- hand limits of $f(x)$ at x i.e. $f_+(x) = \underset{\epsilon \to 0+}{Lt} f(x+\epsilon)$

and $f_-(x) = \underset{\epsilon \to 0-}{Lt} f(x-\epsilon)$

respectively. Dirichlets conditions are sufficient but not necessary and are usually satisfied in most physical problems.

Reference:

1. *Mathematical Methods for Physicists - Arfken & Weber*