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TOPIC- Forward difference interpolation formula.

Interpolation: Let the function $y = f(x)$ is known for $(n+1)$ distinct values of x say, $x_0, x_1, x_2, \ldots, x_n$. There is no other information available about the function $f(x)$. That is

$$f(x_i) = y_i, \quad i = 0, 1, 2, \ldots, n. \quad \text{——— (1)}$$

The problem of interpolation is to compute the value of $f(x)$, atleast approximate, for an argument, say x, not found in the table. i.e., The term INTERPOLATION is used when x lies between the smallest and greatest of the interpolating points $x_0, x_1, x_2, \ldots, x_n$.

Since the explicit form of the function $y = f(x)$ is unknown to us, in general, our problem will be to approximate $f(x)$ by a function $\phi(x)$, which is simple in nature and is such that $\phi(x_i) = f(x_i)$ $(i = 0, 1, 2, \ldots, n)$ ——— (2)

This function $\phi(x)$ is known as the interpolation function and is used to compute the approximate value of the function $f(x)$ as the desired value of x.

In general, $f(x) \approx \phi(x)$ ——→ (3)

If we write $f(x) = \phi(x) + R_{n+1}(x)$ ——→ (4)

then we say that $R_{n+1}(x)$ is the remainder or error committed in replacing $f(x)$ by $\phi(x)$.

This function $\phi(x)$ may be of different types. When $\phi(x)$ is a polynomial we call it a Parabolic or Polynomial Interpolation. When $\phi(x)$ is a sum of exponential functions, we call it an Exponential Interpolation. When $\phi(x)$ is a trigonometric sum then we call it a Trigonometric Interpolation etc.

We shall remain interested in polynomial Interpolation.

** Establish Newton's forward difference interpolation formula for the equispaced interpolating points.

→ Let a function $f(x)$ is known for $(n+1)$ distinct equispaced arguments namely $x_0, x_1, x_2, \ldots, x_n$ such that

$$x_i = x_0 + ih; \quad i = 0, 1, 2, \ldots, n.$$

where h is the length of each space.

It is required to construct a polynomial $\phi(x)$ of degree less than or equal to n satisfying the condition $y_i = \phi(x_i)$, $i = 0, 1, 2, \ldots n$ ——— (1)

Since $\phi(x)$ is a polynomial of degree n, we take $\phi(x)$ as

$$\phi(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) + \cdots + a_n(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n-1}) \quad \text{——— (2)}$$

The constants $a_0, a_1, a_2, \ldots, a_n$ are to be determined successively by using (1).

To determine the values of a_i's, substituting $x = x_i$, $i = 0, 1, 2, \ldots, n$.

putting $x = x_0$ in (2) and using (1) we get

$$\phi(x_0) = a_0 \Rightarrow y_0 = a_0 \Rightarrow a_0 = y_0$$

putting $x = x_1$ in (2) and using (1) we get

$$\phi(x_1) = a_0 + a_1(x_1 - x_0)$$

$$\Rightarrow y_1 = y_0 + a_1 h \Rightarrow a_1 = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{h}$$

putting $x = x_2$ in (2) and using (1) we get

$$\phi(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

$$\Rightarrow y_2 = y_0 + \frac{y_1 - y_0}{h} \cdot 2h + a_2 \cdot 2h \cdot h$$

$$\Rightarrow y_2 = y_0 + 2y_1 - 2y_0 + 2a_2 h^2$$

$$\Rightarrow 2a_2 h^2 = y_2 - 2y_1 + y_0$$

$$\Rightarrow a_2 = \frac{y_2 - 2y_1 + y_0}{2h^2} = \frac{\Delta^2 y_0}{2!\, h^2}$$

Putting $x = x_3$ in (2) and using (1) we get

$$\phi(x_3) = a_0 + a_1(x_3 - x_0) + a_2(x_3 - x_0)(x_3 - x_1) + a_3(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)$$

$$\Rightarrow y_3 = y_0 + \frac{y_1 - y_0}{h}\cdot(3h) + \frac{y_2 - 2y_1 + y_0}{2h^2}\cdot(3h)(2h) + a_3\cdot(3h)\cdot(2h)\cdot(h)$$

$$\Rightarrow y_3 = y_0 + 3y_1 - 3y_0 + 3y_2 - 6y_1 + 3y_0 + 6a_3h^3$$

$$\Rightarrow 6a_3h^3 = +y_0 - 3y_1 + 3y_2 = y_3$$

$$\Rightarrow a_3 = \frac{y_3 - 3y_2 + 3y_1 - y_0}{6h^3} = \frac{\Delta^3 y_0}{3!\,h^3}$$

Similiarly calculating the coefficients we shall find that

$$a_4 = \frac{\Delta^4 y_0}{4!\,h^4}, \quad a_5 = \frac{\Delta^5 y_0}{5!\,h^5}, \ldots\ldots, a_n = \frac{\Delta^n y_0}{n!\,h^n}$$

Using these values, (2) becomes

$$\phi(x) = y_0 + (x - x_0)\frac{\Delta y_0}{h} + (x - x_0)(x - x_1)\frac{\Delta^2 y_0}{2!\,h^2} + (x - x_0)(x - x_1)(x - x_2)\frac{\Delta^3 y_0}{3!\,h^3}$$
$$+ \cdots + (x - x_0)(x - x_1)\cdots(x - x_{n-1})\frac{\Delta^n y_0}{n!\,h^n} \quad \text{——— (3)}$$

Introducing the condition $x_i = x_0 + ih$, $i = 0, 1, 2, \ldots, n$ for equispaced points and a new variable u as

$$x = x_0 + uh \Rightarrow x_0 = x - uh$$

Therefore $x_i = x_0 + ih \Rightarrow x_i = x - uh + ih$

$$\Rightarrow x_i - x = (i - u)h$$

$$\Rightarrow x - x_i = (u - i)h, \quad i = 0, 1, 2, \ldots n.$$

From (3) we get

$$\phi(x) = y_0 + (uh)\frac{\Delta y_0}{h} + (uh)(u-1)h\frac{\Delta^2 y_0}{2!\,h^2} + \cancel{(uh)(u-1)h(u-2)h}$$
$$+ (uh)(u-1)h(u-2)h\frac{\Delta^3 y_0}{3!\,h^3} + \cdots + (uh)(u-1)h(u-2)h$$
$$\cdots(u - \overline{n-1})h\frac{\Delta^n y_0}{n!\,h^n}$$

$$= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$
$$+ \frac{u(u-1)(u-2)\cdots(u - \overline{n-1})}{n!}\Delta^n y_0, \text{ where } u = \frac{x - x_0}{h}$$

This is known as Newton's forward or Newton-Gregory difference interpolating polynomial.

⊛ Find the polynomial of degree ≤ 3 passing through the points $(-1,1)$, $(0,1)$, $(1,1)$, and $(2,3)$.

→ The forward difference table is:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
-1	1			
		0		
0	1		0	
		0		2
1	1		2	
		2		
2	3			

Newton forward interpolation formula is

$$f(x) = y = y_0 + u \cdot \Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

where $u = \dfrac{x - x_0}{h}$

Here $x_0 = -1$, $h = 1$.

$\therefore \quad u = \dfrac{x+1}{1} = x+1$

$$\therefore \quad y = 1 + (x+1)\cdot 0 + \frac{x(x+1)}{2}\cdot 0 + \frac{x(x+1)(x-1)}{6}\cdot 2$$

$$= 1 + \frac{(x^2+x)(x-1)}{3}$$

$$= 1 + \frac{x^3 - x^2 + x^2 - x}{3}$$

$$= 1 + \frac{x^3 - x}{3}$$

$$= \tfrac{1}{3}(x^3 - x + 3) \longrightarrow \underline{\text{Ans}}$$

⊛ Estimate u_2 from the following table:

x :	1	2	3	4	5
u_x :	7	-	13	21	37

⇒ Since the four values of u_x are given. Therefore u_x is a polynomial of degree 3.

∴ 4th differences of u_x vanishes.

∴ $\Delta^4 u_1 = 0$ for $u = 1$

$\Rightarrow (E-1)^4 u_1 = 0$

$\Rightarrow (E^4 - {}^4C_1 E^3 + {}^4C_2 E^2 - {}^4C_3 E + 1) u_1 = 0$

$\Rightarrow E^4(u_1) - 4E^3(u_1) + 6E^2(u_1) - 4E(u_1) + u_1 = 0$

$\Rightarrow u_5 - 4u_4 + 6u_3 - 4u_2 + u_1 = 0$

$\Rightarrow 37 - 4\times 21 + 6\times 13 - 4E_2 + 7 = 0$

$\Rightarrow 38 - 4u_2 = 0 \Rightarrow u_2 = \frac{38}{4} = 9.5$

Therefore the required values of u_2 is 9.5.

⊛ If $y(0) = -1$, $y(1) = 3$, $y(2) = 8$, $y(3) = 13$, $y(6) = ?$

⇒ The difference table is:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	-1			
		4		
1	3		1	
		5		-1
2	8		0	
		5		
3	13			

$\therefore y(6) = y(0 + 6\cdot 1)$

$= E^6 y_0$

$= (1+\Delta)^6 y_0$

$= (1 + {}^6C_1 \Delta + {}^6C_2 \Delta^2 + {}^6C_3 \Delta^3 + \cdots) y_0$

$= y_0 + 6\Delta y_0 + 15\Delta^2 y_0 + 20\Delta^3 y_0 + \cdots$

$= -1 + 6\times 4 + 15\times 1 + 20\times(-1)$

$= -1 + 24 + 15 - 20 = 18$