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Dr. Tapanendu Kamilya

Assistant Professor, Department of Physics, Narajole Raj College

Topic:

Properties of Magnetic Field: Curl and divergence. Axial vector property of B and its consequences. Vector Potential. Magnetic Force on (1) point charge (2) current carrying wire (3) between current elements. Torque on a current loop in a uniform Magnetic Field.

Magnetostatics

The Divergence of Magnetic Field:

According to Biot- Savart Law the magnetic field B at any point P (x, y, z) due to a volume distribution of current J(x', y', z') is-

$$\vec{B} = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J} \times \vec{R}}{R^3} dV$$

Where, $dV = dx' dy' dz'$

$$\vec{\nabla} \cdot \vec{B} = \frac{\mu_0}{4\pi} \int_V \vec{\nabla} \cdot \left(\frac{\vec{J} \times \vec{R}}{R^3} \right) dV$$

$$\vec{\nabla} \cdot \left(\frac{\vec{J} \times \vec{R}}{R^3} \right) = \frac{\vec{R}}{R^3} \cdot (\vec{\nabla} \times \vec{J}) - \vec{J} \cdot (\vec{\nabla} \times \frac{\vec{R}}{R^3})$$

$$\vec{\nabla} \times \frac{\vec{R}}{R^3} = \frac{1}{R^3} (\vec{\nabla} \times \vec{R}) + \vec{\nabla} \frac{1}{R^3} \times \vec{R}$$

$$\text{Here, } \vec{\nabla} \times \vec{R} = 0, \text{ and } \vec{\nabla} \frac{1}{R^3} = -\frac{3\vec{R}}{R^5}$$

$$\text{Thus, } \vec{J} \cdot (\vec{\nabla} \times \frac{\vec{R}}{R^3}) = 0$$

$$\text{Hence, } \vec{\nabla} \cdot \vec{B} = 0$$

This result Gauss's Law in Magnetostatics, i. e. $\int_V \vec{\nabla} \cdot \vec{B} dV = \oint_S \vec{B} \cdot d\vec{S} = 0$

The Curl of Magnetic Field:

$$\vec{B} = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J} \times \vec{R}}{R^3} dV \dots \dots \dots (1)$$

Where, $dV = dx' dy' dz'$ and $\vec{\nabla} \frac{1}{R} = -\frac{\vec{R}}{R^3}$



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$$\vec{B} = -\frac{\mu_0}{4\pi} \int \vec{j} \times \vec{\nabla} \frac{1}{R} dV \dots\dots\dots(2)$$

$$\vec{\nabla} \times \frac{1}{R} \vec{j} = \frac{1}{R} \vec{\nabla} \times \vec{j} + \vec{\nabla} \frac{1}{R} \times \vec{j} = -\vec{j} \times \vec{\nabla} \frac{1}{R}$$

Therefore, $\vec{B} = \frac{\mu_0}{4\pi} \vec{\nabla} \times \int \frac{1}{R} \vec{j} dV$

We know that $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$

$$\vec{\nabla} \times \vec{B} = \frac{\mu_0}{4\pi} \vec{\nabla} (\vec{\nabla} \cdot \int \frac{1}{R} \vec{j} dV) - \frac{\mu_0}{4\pi} \nabla^2 \int \frac{1}{R} \vec{j} dV$$

$$= \frac{\mu_0}{4\pi} \vec{\nabla} \int \vec{j} \cdot \vec{\nabla} \frac{1}{R} dV - \frac{\mu_0}{4\pi} \int \vec{j} \nabla^2 \frac{1}{R} dV$$

$$\nabla^2 \frac{1}{R} = -4\pi\delta(\vec{r} - \vec{r}') \text{ and } \vec{\nabla} \frac{1}{R} = -\vec{\nabla}' \frac{1}{R}$$

Therefore, $\vec{\nabla} \times \vec{B} = -\frac{\mu_0}{4\pi} \int \vec{j} \cdot \vec{\nabla}' \frac{1}{R} dV + \frac{\mu_0}{4\pi} \int \vec{j} 4\pi\delta(\vec{r} - \vec{r}') dV$

$$= -\frac{\mu_0}{4\pi} \vec{\nabla} \int [\vec{\nabla}' \cdot \frac{1}{R} \vec{j} - \frac{1}{R} \vec{\nabla}' \cdot \vec{j}] dV + \mu_0 \vec{j}(\vec{r})$$

For steady current

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}(\vec{r}) - \frac{\mu_0}{4\pi} \vec{\nabla} \int [\vec{\nabla}' \cdot \frac{1}{R} \vec{j}] dV$$

The integral on the right hand side can be converted using divergence theorem to a surface integral.

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}(\vec{r})$$

$$\int_S (\vec{\nabla} \times \vec{B}) \cdot \vec{dS} = \mu_0 \int_S \vec{j} \cdot \vec{dS}$$

Or, $\oint \vec{B} \cdot \vec{dl} = \mu_0 I$

Magnetic Vector Potential:

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{j}$$

Or, $\vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{j}$

We chose that $\vec{\nabla} \cdot \vec{A} = 0$ and $\nabla^2 \vec{A} = -\mu_0 \vec{j}$ is called vector Poisson's equation.



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$$\varphi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho dV}{R} \dots\dots\dots(1)$$

$$\vec{A}(\vec{r}') = \frac{\mu_0}{4\pi} \int \frac{\vec{J} dV}{R} \dots\dots\dots(2)$$

The solution of (2) can be obtained from $\vec{B} = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J} \times \vec{R}}{R^3} dV \dots\dots(3)$

$$\vec{\nabla} \frac{1}{R} = -\frac{\vec{R}}{R^3} \text{ and } \vec{\nabla} \times \frac{1}{R} \vec{J} - \frac{1}{R} \vec{\nabla} \times \vec{J}$$

$$\vec{J} \times \frac{\vec{R}}{R^3} = \vec{\nabla} \times \frac{1}{R} \vec{J} \dots\dots\dots(4)$$

Therefore, from equn. (3) and (4)

$$\vec{B} = \vec{\nabla} \times \frac{\mu_0}{4\pi} \int \frac{\vec{J}}{R} dV$$

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}}{R} dV$$

Derivation of Biot-Savart Law and Ampere's Law from the Concept of Magnetic Vector Potential:

Biot-Savart Law:

The magnetic vector potential $\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}}{R} dV$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{\mu_0}{4\pi} \int \vec{\nabla} \times \frac{\vec{J}}{R} dV$$

$$\vec{\nabla} \times \frac{1}{R} \vec{J} = \frac{1}{R} \vec{\nabla} \times \vec{J} + \vec{\nabla} \frac{1}{R} \times \vec{J} = -\vec{J} \times \vec{\nabla} \frac{1}{R} = -\frac{\vec{R}}{R^3} \times \vec{J}$$

Therefore, $\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \times \vec{R}}{R^3} dV$

And each volume current element $\vec{J} dV$ contributes the quantity-

$$d\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \times \vec{R}}{R^3} dV$$

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I d\vec{l} \times \vec{R}}{R^3}$$

This is known as *Biot – Savart Law*.

Ampere's Law:



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$$\oint_C \vec{B} \cdot d\vec{l} = \oint_C (\vec{\nabla} \times \vec{A}) \cdot d\vec{l} = \iint \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) \cdot d\vec{S}$$

$$\text{We know that } \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

$$\text{We choose, } \vec{\nabla} \cdot \vec{A} = 0$$

$$\oint_C \vec{B} \cdot d\vec{l} = \iint \mu_0 \vec{J} \cdot d\vec{S} = \mu_0 I, \text{ which is Ampere's circuital Law.}$$

Calculation of A for straight current carrying conductor:

We consider a straight wire AB carrying a steady current I. We want to calculate vector potential \vec{A} at any point P. We consider an element $d\vec{l} = z dz$

$$\vec{A} = \frac{\mu_0 I d}{4\pi R}$$

$$\vec{A} = z \frac{\mu_0 I}{4\pi} \int_{-l_1}^{+l_2} \frac{dz}{\sqrt{r^2 + z^2}}$$

$$\vec{A} = z \frac{\mu_0 I}{4\pi} [\ln(z + \sqrt{r^2 + z^2})]_{-l_1}^{+l_2}$$

$$\vec{A} = z \frac{\mu_0 I}{4\pi} \ln \left[\frac{L + \sqrt{r^2 + L^2}}{-L + \sqrt{r^2 + L^2}} \right]$$

$$\vec{A} = z \frac{\mu_0 I}{4\pi} \ln \left[\frac{1 + (1 + \frac{r^2}{L^2})^{\frac{1}{2}}}{-1 + (1 + \frac{r^2}{L^2})^{\frac{1}{2}}} \right]$$

$$\vec{A} = z \frac{\mu_0 I}{4\pi} \ln \left[\frac{2 + \frac{r^2}{2L^2}}{\frac{r^2}{2L^2}} \right]$$

$$\vec{A} = z \frac{\mu_0 I}{4\pi} \ln \left[\frac{4L^2}{r^2} + 1 \right]$$

$$\vec{A} = z \frac{\mu_0 I}{4\pi} \ln \left(\frac{2L}{r} \right)$$

We know that

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$= \frac{1}{r} \begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & 0 & A_z \end{vmatrix} = -\hat{\theta} \frac{\partial A_z}{\partial r} = -\hat{\theta} \frac{\partial}{\partial r} \left[\frac{\mu_0 I}{2\pi} \ln \left(\frac{2L}{r} \right) \right]$$

$$= \hat{\theta} \frac{\mu_0 I}{2\pi r}$$



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2. Two long straight parallel wires carrying same current:

Suppose we have two long parallel wires each of length $2L$ carrying current I in opposite directions. The magnetic vector potential $\vec{A} = A_1 + A_2$

$$= z \frac{\mu_0 I}{2\pi} \ln\left(\frac{2L}{r_1}\right) - z \frac{\mu_0 I}{2\pi} \ln\left(\frac{2L}{r_2}\right)$$

$$= z \frac{\mu_0 I}{2\pi} \ln\left(\frac{r_2}{r_1}\right)$$

Here, r_1 and r_2 are the distances of the point of observation P from the wires and z is a unit vector parallel to the wires.

Torque acting in small current loop:

Force on a current element $I d\vec{l} = d\vec{F} = I d\vec{l} \times \vec{B}$

$$d\vec{\tau} = \vec{r} \times d\vec{F} = d\vec{F} = I \vec{r} \times (d\vec{l} \times \vec{B})$$

From vector analysis –

$$\vec{r} \times (d\vec{l} \times \vec{B}) + d\vec{l} \times (\vec{B} \times \vec{r}) + \vec{B} \times (\vec{r} \times d\vec{l}) = 0$$

$$\vec{r} \times (d\vec{l} \times \vec{B}) = -d\vec{l} \times (\vec{B} \times \vec{r}) - \vec{B} \times (\vec{r} \times d\vec{l})$$

We know that-

$$d[\vec{r} \times (\vec{r} \times \vec{B})] = d\vec{r} \times (\vec{r} \times \vec{B}) + \vec{r} \times (d\vec{r} \times \vec{B})$$

Since, $d\vec{r} = d\vec{l}$ and $\vec{B} = \text{uniform}$

$$\vec{r} \times (d\vec{l} \times \vec{B}) = d[\vec{r} \times (\vec{r} \times \vec{B})] - d\vec{l} \times (\vec{r} \times \vec{B})$$

$$2\vec{r} \times (d\vec{l} \times \vec{B}) = d[\vec{r} \times (\vec{r} \times \vec{B})] - \vec{B} \times (\vec{r} \times d\vec{l})$$

Therefore, $d\vec{\tau} = \frac{1}{2} I \{d[\vec{r} \times (\vec{r} \times \vec{B})] - \vec{B} \times (\vec{r} \times d\vec{l})\}$

Total torque on the loop is $d\vec{\tau} = \frac{1}{2} I \{d[\vec{r} \times (\vec{r} \times \vec{B})] - \vec{B} \times \phi(\vec{r} \times d\vec{r})\}$



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$$\vec{\tau} = \frac{1}{2} I \oint (\vec{r} \times d\vec{r}) \times \vec{B}$$

$$\vec{\tau} = \vec{m} \times \vec{B}$$

$$\vec{m} = \frac{1}{2} I \oint (\vec{r} \times d\vec{r})$$

Frequently Asked Questions & Numerical:

Theoretical questions, Solved Problems and Problems in exercise of Foundations of Electricity and Magnetism-Author: Basudev Ghosh, Books And Allied Publications, 2018 (Edn)

References

- (i) *Foundations of Electricity and Magnetism-Author: Basudev Ghosh, Books And Allied Publications, 2018 (Edn)*
- (ii) *Electricity And Magnetism-Rakshit and Chattopaddhayay*

Link to Audio visual Lectures (e-Lectures) on this topic given by Distinguish Professors of Indian & Foreign Universities:

- (i) <https://www.youtube.com/watch?v=ra1BUT4p654>
- (ii) <https://www.youtube.com/watch?v=0m0B131LOmU>
- (iii) <https://www.youtube.com/watch?v=2MSsNcpYeb8>
- (iv) <https://www.youtube.com/watch?v=2QCt7ZbNOAU>
- (v) <https://nptel.ac.in/courses/115101005/>