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Topic: Integral Transforms

An integral transform maps an equation from its original domain into another domain where it might be manipulated and solved much more easily than in the original domain. The solution is then mapped back to the original domain using the inverse of the integral.

The motivation behind integral transforms is easy to understand. There are many classes of problems that are difficult to solve in their original representations. An integral transform "maps" an equation from its original "domain" into another domain. Manipulating and solving the equation in the target domain can be much easier than manipulation and solution in the original domain. The solution is then mapped back to the original domain with the inverse of the integral transform.

The precursor of the transforms were the Fourier series to express functions in finite intervals. Later the Fourier transform was developed to remove the requirement of finite intervals. So, before we go on to learn integral transforms further, we will briefly discuss Fourier Series in the following section.

1.1 Fourier series

If a function $h(t)$, which varies with t, satisfies *the Dirichlet conditions*

1. $h(t)$ is defined from $t = -\infty$ to $t = +\infty$ and is periodic with some period T,
2. $h(t)$ is well-defined and single-valued (except possibly in a finite number of points) in the interval $\left[-\frac{1}{2}T, \frac{1}{2}T\right]$,
3. $h(t)$ and its derivative $\mathrm{d}h(t)/\mathrm{d}t$ are continuous (except possibly in a finite number of step discontinuities) in the interval $\left(-\frac{1}{2}T, \frac{1}{2}T\right)$, and

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4. $h(t)$ is absolutely integrable in the interval $\left[-\frac{1}{2}T, \frac{1}{2}T\right]$, that is, $\int_{-T/2}^{T/2} |h(t)|\,\mathrm{d}t < \infty$,

then the function $h(t)$ can be expressed as a *Fourier series* expansion

$$h(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)] = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}, \tag{1.1}$$

where

$$\begin{cases} \omega_0 &= \dfrac{2\pi}{T} = 2\pi f_0, \\ a_n &= \dfrac{2}{T}\displaystyle\int_{-T/2}^{T/2} h(t)\cos(n\omega_0 t)\,\mathrm{d}t, \\ b_n &= \dfrac{2}{T}\displaystyle\int_{-T/2}^{T/2} h(t)\sin(n\omega_0 t)\,\mathrm{d}t, \\ c_n &= \dfrac{1}{T}\displaystyle\int_{-T/2}^{T/2} h(t)e^{-in\omega_0 t}\,\mathrm{d}t = \dfrac{1}{2}(a_n - ib_n), \\ c_{-n} &= c_n^* = \dfrac{1}{2}(a_n + ib_n). \end{cases} \tag{1.2}$$

f_0 is called the *fundamental frequency* of the system. In the Fourier series, a function $h(t)$ is analyzed into an infinite sum of harmonic components at multiples of the fundamental frequency. The coefficients a_n, b_n and c_n are the *amplitudes* of these harmonic components.

At every point where the function $h(t)$ is continuous the Fourier series converges uniformly to $h(t)$. If the Fourier series is truncated, and $h(t)$ is approximated by a sum of only a finite number of terms of the Fourier series, then this approximation differs somewhat from $h(t)$. Generally, the approximation becomes better and better as more and more terms are included.

At every point $t = t_0$ where the function $h(t)$ has a step discontinuity the Fourier series converges to the average of the limiting values of $h(t)$ as the point is approached from above and from below:

$$\left[\lim_{\varepsilon\to 0+} h(t_0+\varepsilon) + \lim_{\varepsilon\to 0+} h(t_0-\varepsilon)\right] \Big/ 2.$$

Around a step discontinuity, a truncated Fourier series overshoots at both sides near the step, and oscillates around the true value of the function $h(t)$. This oscillation behavior in the vicinity of a point of discontinuity is called the *Gibbs phenomenon*.

The coefficients c_n in Equation 1.1 are the *complex amplitudes* of the harmonic components at the frequencies $f_n = nf_0 = n/T$. The complex amplitudes c_n as a function of the corresponding frequencies f_n constitute a *discrete complex amplitude spectrum*.

Example 1.1: Examine the Fourier series of the *square wave* shown in Figure 1.1.

Solution. Applying Equation 1.2, the square wave can be expressed as the Fourier series

$$\begin{aligned} h(t) &= \frac{4}{\pi}\left[\cos(\omega_0 t) - \frac{1}{3}\cos(3\omega_0 t) + \frac{1}{5}\cos(5\omega_0 t) - \cdots\right] \\ &= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\pi/2)}{n}\cos(n\omega_0 t). \end{aligned}$$

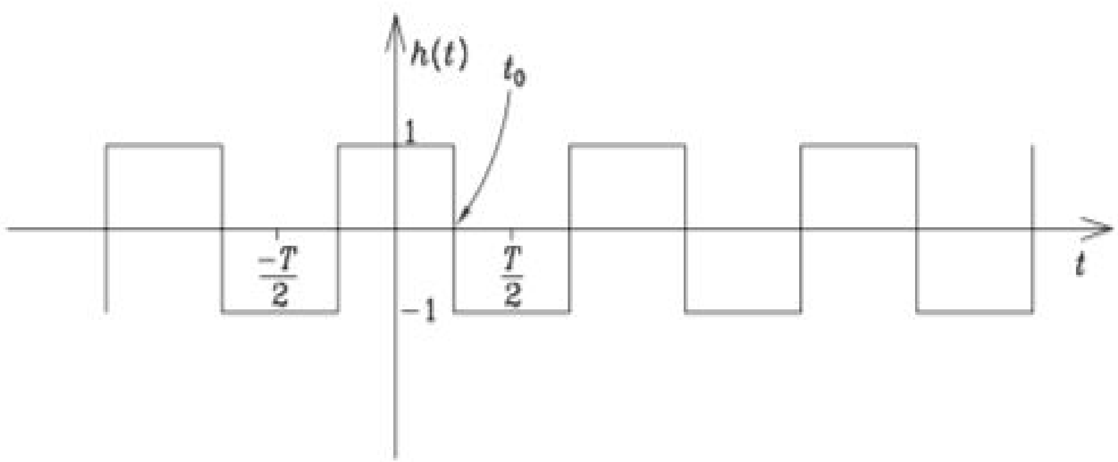

Figure 1.1: Square wave $h(t)$.

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If this Fourier series is truncated, and the function is approximated by a finite sum, then this approximation differs from the original square wave, especially around the points of discontinuity. Figure 1.2 illustrates the Gibbs oscillation around the point $t = t_0$ of the square wave of Figure 1.1.

The amplitude spectrum of the square wave of Figure 1.1 is shown in Figure 1.3. The amplitude coefficients of the square wave are $c_n = \frac{1}{2} a_n = 0, \frac{2}{\pi}, 0, -\frac{2}{3\pi}, 0, \frac{2}{5\pi}, 0, \ldots$

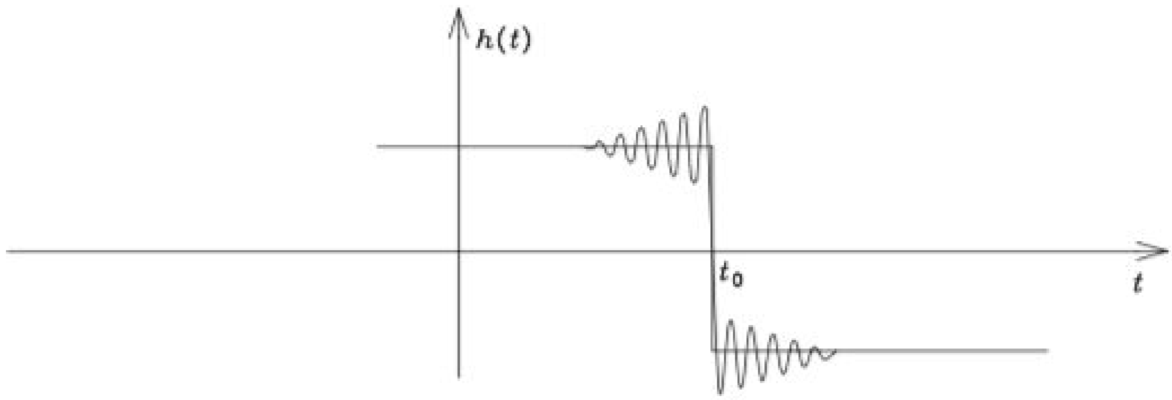

Figure 1.2: The principle how the truncated Fourier series of the square wave $h(t)$ of Fig. 1.1 oscillates around the true value in the vicinity of the point of discontinuity $t = t_0$.

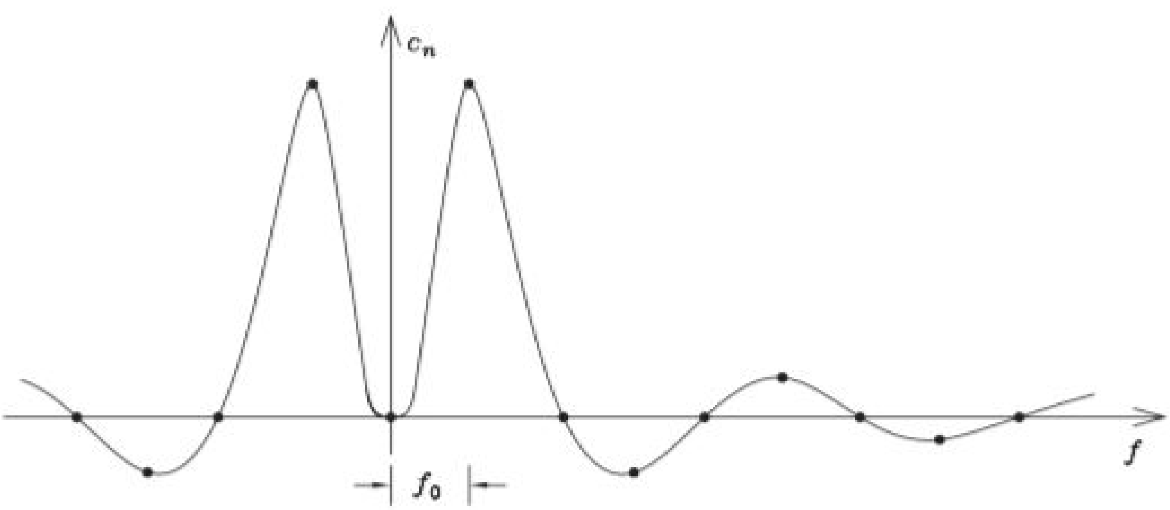

Figure 1.3: Discrete amplitude spectrum of the square wave $h(t)$ of Fig. 1.1, formed by the amplitude coefficients c_n. f_0 is the fundamental frequency.

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1.2 Fourier transform

In the previous section, we have learnt the Fourier series representation of a periodic function in a fixed interval as a superposition of sinusoidal functions. It is often desirable, however, to obtain such a representation even for functions defined over an infinite interval and with no particular periodicity. Such a representation is called a **Fourier transform** and is one of a class of representations called integral transforms.

We begin by considering Fourier transforms as a generalisation of Fourier series. We will then discuss the properties of the Fourier transform and its applications.

The Fourier series, Equation 1.1, can be used to analyze periodic functions of a period T and a fundamental frequency $f_0 = \frac{1}{T}$. By letting the period tend to infinity and the fundamental frequency to zero, we can obtain a generalization of the Fourier series which also is suitable for analysis of non-periodic functions.

According to Equation 1.1,

$$h(t) = \sum_{n=-\infty}^{\infty} \underbrace{\frac{1}{T} \int_{-T/2}^{T/2} h(t')e^{-i2\pi n f_0 t'}\, dt'}_{c_n} e^{i12\pi n f_0 t}. \tag{1.3}$$

We shall replace nf_0 by f, and let $T \to \infty$ and $1/T = f_0 = df \to 0$. In this case,

$$\sum_{n=-\infty}^{\infty} \frac{1}{T} \to \int_{-\infty}^{\infty} df, \tag{1.4}$$

and

$$h(t) = \int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{\infty} h(t')e^{-i2\pi f t'}\, dt' \right]}_{H(f)} e^{i2\pi f t}\, df. \tag{1.5}$$

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We can interpret this formula as the sum of the waves $H(f)\,\mathrm{d}f\,e^{i2\pi ft}$.

With the help of the notation $H(f)$, we can write Equation 1.5 in the compact form

$$h(t) = \int_{-\infty}^{\infty} H(f)e^{i2\pi ft}\,\mathrm{d}f = \mathcal{F}\{H(f)\}. \tag{1.6}$$

The operation $\mathcal{F}$ is called the *Fourier transform*. From above,

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-i2\pi ft}\,\mathrm{d}t = \mathcal{F}^{-1}\{h(t)\}. \tag{1.7}$$

The operation $\mathcal{F}^{-1}$ is called the *inverse Fourier transform*.

Functions $h(t)$ and $H(f)$ which are connected by Equations 1.6 and 1.7 constitute a *Fourier transform pair.* Notice that even though we have used as the variables the symbols t and f, which often refer to time [s] and frequency [Hz], the Fourier transform pair can be formed for *any variables, as long as the product of their dimensions is one* (the dimension of one variable is the inverse of the dimension of the other).

In the literature, it is possible to find several, slightly differing ways to define the Fourier integrals. They may differ in the constant coefficients in front of the integrals and in the exponents. Here, we have chosen the definitions in Equations 1.6 and 1.7, because they are the most convenient for our purposes. *In our definition, the exponential functions inside the integrals carry the coefficient* 2π*, because, in this way, we can avoid the coefficients in front of the integrals.* We have noticed that coefficients in front of Fourier integrals are a constant source of mistakes in calculations, and, by our definition, these mistakes can be avoided. Also the theorems of Fourier transform are essentially simpler, if this definition is chosen: in this way even they, except the derivative theorem, have no front coefficients.

Table 1.1 lists a few important Fourier transform pairs, which will be useful in this book, as well as the full width at half maximum, FWHM, of these functions.

Table 1.1: Fourier transform pairs $h(t)$ and $H(f)$, and the FWHM of these functions.

Name of $h(t)$	$h(t)$	FWHM of $h(t)$	$\mathcal{F}^{-1}, \mathcal{F}$ $\Leftrightarrow$	Name of $H(t)$	$H(f)$	FWHM of $H(f)$
boxcar	$\Pi_{2T}(t) = \begin{cases} 1, & \|t\| \le T, \\ 0, & \|t\| > T \end{cases}$	$2T$		sinc	$2T\,\mathrm{sinc}(\pi 2T f)$	$\frac{1.2067}{2T}$
triangular	$\Lambda_T(t) = \begin{cases} 1 - \frac{\|t\|}{T}, & \|t\| \le T, \\ 0, & \|t\| > T \end{cases}$	T		sinc^2	$T\,\mathrm{sinc}^2(\pi T f)$	$\frac{1.7718}{2T}$
Lorentzian	$\frac{\sigma/\pi}{\sigma^2 + t^2}$	2σ		exponential	$\exp(-\pi 2\sigma \|f\|)$	$\frac{\ln 2}{\pi\sigma}$
Gaussian	$\sqrt{\frac{\alpha}{\pi}}\exp(-\alpha t^2)$	$2\sqrt{\frac{\ln 2}{\alpha}}$		Gaussian	$\exp\left(\frac{-\pi^2 f^2}{\alpha}\right)$	$\frac{2}{\pi}\sqrt{\alpha \ln 2}$
Dirac's delta	$\delta(t - t_0)$	0	$\mathcal{F}$ $\Rightarrow$	exponential wave	$\exp(i2\pi t_0 f)$	—
Dirac's delta	$\delta(t - t_0)$	0	$\mathcal{F}^{-1}$ $\Rightarrow$	exponential wave	$\exp(-i2\pi t_0 f)$	—

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Example 1.2: Applying Fourier transforms, compute the integral $\int_0^\infty \frac{\sin(px)\cos(qx)}{x}\,dx$, where $p > 0$ and $p \neq q$.

Solution. Knowing that the imaginary part of e^{iqx} is antisymmetric, we can write

$$\begin{aligned}\int_0^\infty \frac{\sin(px)\cos(qx)}{x}\,dx &= \frac{p}{2}\int_{-\infty}^{\infty} \frac{\sin(px)}{px} e^{iqx}\,dx = \frac{p}{2}\int_{-\infty}^{\infty} \mathrm{sinc}(px) e^{i2\pi \frac{q}{2\pi} x}\,dx \\ &= \frac{\pi}{2} h\left(\frac{q}{2\pi}\right),\end{aligned}$$

where the function

$$h(t) = \mathcal{F}\left\{\frac{p}{\pi}\,\mathrm{sinc}\left(\pi \frac{p}{\pi} f\right)\right\}.$$

From Table 1.1, we know that the Fourier transform of a sinc function is a boxcar function. Consequently,

$$\frac{\pi}{2} h\left(\frac{q}{2\pi}\right) = \frac{\pi}{2}\Pi_{p/\pi}\left(\frac{q}{2\pi}\right) = \begin{cases} \pi/2, & |q| < p, \\ 0, & |q| > p. \end{cases}$$

1.3 Dirac's delta function

Dirac's delta function, $\delta(t)$, also called the *impulse function*, is a concept which is frequently used to describe quantities which are localized in one point. Even though real physical quantities cannot be truly localized in exactly one point, the concept of Dirac's delta function is very useful.

Dirac's delta function is defined with the following equation:

$$\int_{-\infty}^{\infty} F(t)\delta(t - t_0)\,dt = F(t_0), \tag{1.10}$$

where $F(t)$ is an arbitrary function of t, continuous at the point $t = t_0$.

By inserting the function $F(t) \equiv 1$ in Equation 1.10, we can see that the area of Dirac's delta function is equal to unity, that is,

$$\int_{-\infty}^{\infty} \delta(t - t_0)\,\mathrm{d}t = 1. \tag{1.11}$$

In the usual sense, $\delta(t)$ is not really a function at all. In practice,

$$\lim_{t \to t_0} \delta(t - t_0) = \infty. \tag{1.12}$$

At points $t \neq t_0$ either $\delta(t - t_0) = 0$ or $\delta(t)$ oscillates with infinite frequency.

It can be shown that Dirac's delta function has the following properties:

$$\begin{cases} \delta(-t) & \triangleq & \delta(t), \\ t\delta(t) & \triangleq & 0, \\ \delta(at) & \triangleq & \dfrac{1}{a}\delta(t), \quad \text{if } a > 0, \\ \dfrac{\mathrm{d}\delta(t)}{\mathrm{d}t} & \triangleq & -\dfrac{1}{t}\delta(t), \\ F(t)\delta(t - t_0) & \triangleq & F(t_0)\delta(t - t_0), \end{cases} \tag{1.13}$$

where the correspondence relation $\triangleq$ means that the value of the integral in Equation 1.10 remains the same whichever side of the relation is inserted in the integral.

The "shape" of Dirac's delta function is not uniquely defined. There are infinitely many representations of $\delta(t)$ which satisfy Equation 1.10. One of them, very useful with Fourier transforms, is

$$\boxed{\delta(t) = \int_{-\infty}^{\infty} e^{i2\pi ts}\,\mathrm{d}s = \mathcal{F}\{1\} = \int_{-\infty}^{\infty} \cos(2\pi ts)\,\mathrm{d}s.} \tag{1.14}$$

Equivalently, we can write

$$\boxed{\delta(t) = \int_{-\infty}^{\infty} e^{-i2\pi ts}\,\mathrm{d}s = \mathcal{F}^{-1}\{1\} = \int_{-\infty}^{\infty} \cos(2\pi ts)\,\mathrm{d}s.} \tag{1.15}$$

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A few other useful representations for Dirac's delta function are, for example,

$$\delta(t) \triangleq \lim_{a\to\infty} \frac{\sin(a\pi t)}{\pi t} \triangleq \lim_{\sigma\to 0+} \frac{\sigma/\pi}{t^2+\sigma^2}$$

$$\triangleq \lim_{a\to\infty} \frac{a}{2} e^{-a|t|} \triangleq \lim_{a\to 0+} \frac{1}{a\sqrt{2\pi}} e^{-t^2/(2a^2)}. \qquad (1.16)$$

Two rather simple representations are

$$\delta(t) \triangleq \lim_{a\to 0+} f(t,a) \triangleq \lim_{a\to 0+} g(t,a), \qquad (1.17)$$

where $f(t,a)$ and $g(t,a)$ are the functions illustrated in Figure 1.4.

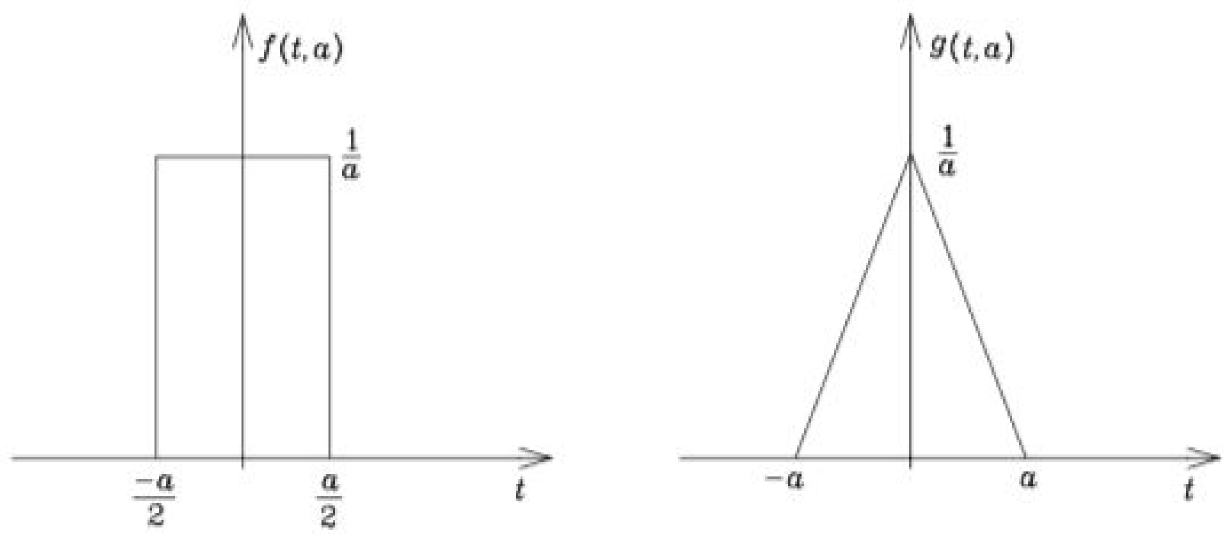

Figure 1.4: Two representations for Dirac's delta function: $\delta(t) \triangleq \lim_{a\to 0+} f(t,a) \triangleq \lim_{a\to 0+} g(t,a)$.

References

1. *Mathematical Methods for Physicists: A Concise Introduction - Tai L. Chow*
2. *Fourier Transforms in Spectroscopy - Jyrki Kauppinen, Jari Partanen*