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C13T (Electromagnetic Theory)

Topic – Polarization of Electromagnetic Waves (Part – 2)

We have already discussed part 1 of this e-report.

Now let us continue part 2 of it.

Fresnel's Formula:

Using Maxwell's two equations $\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$ and $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ we obtain the following two equations: $n\hat{s} \times \vec{H} = -\vec{D}$ and $n\hat{s} \times \vec{E} = -\mu\vec{H}$ where the relation $\vec{B} = \mu\vec{H}$ has been used. Here n is the refractive index of the medium and $\hat{s}$ (known as the wave normal) is the unit vector along the direction of wave vector $\vec{k}$.

Eliminating $\vec{H}$ from the equations and using some vector identities we obtain $\mu\vec{D} = -n^2\hat{s} \times (\hat{s} \times \vec{E}) = n^2[\vec{E} - \hat{s}(\hat{s}.\vec{E})]$.

We can write from here $\mu\varepsilon\vec{E} = n^2[\vec{E} - \hat{s}(\hat{s}.\vec{E})]$, since $\vec{D} = \varepsilon\vec{E}$.

In terms of the components this equation can be written as $\mu\varepsilon_k E_k = n^2[E_k - s_k(\hat{s}.\vec{E})]$ for $k = x, y, z$.

Eliminating the electric field from these 3 sets of equations we finally get

$$\frac{{s_x}^2}{n^2-\mu\varepsilon_x} + \frac{{s_y}^2}{n^2-\mu\varepsilon_y} + \frac{{s_z}^2}{n^2-\mu\varepsilon_z} = \frac{1}{n^2}$$

This is known as the *Fresnel's Formula* for the propagation of light.

Here we define the three principal velocities of propagation by $v_x = \frac{c}{\sqrt{\mu\varepsilon_x}}$, $v_y = \frac{c}{\sqrt{\mu\varepsilon_y}}$ and $v_z = \frac{c}{\sqrt{\mu\varepsilon_z}}$. We also have the phase velocity $v_p = \frac{c}{n}$.

Then the Fresnel's Formula can be written as (very useful form)

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$$s_x{}^2(v_p{}^2 - v_y{}^2)(v_p{}^2 - v_z{}^2) + s_y{}^2(v_p{}^2 - v_z{}^2)(v_p{}^2 - v_x{}^2) + s_z{}^2(v_p{}^2 - v_x{}^2)(v_p{}^2 - v_y{}^2) = 0$$

Optical Classification of Crystals, Uniaxial and Biaxial Crystals:

Transparent crystals fall into three distinct groups as regards their optical properties.

Group I. Crystals in which three crystallographically-equivalent, mutually-orthogonal directions may be chosen. These are crystals of the so-called cubic system. The equivalent directions evidently coincide with the principal dielectric axes and one has $\varepsilon_x = \varepsilon_y = \varepsilon_z = \varepsilon$ (say), then $\vec{D} = \varepsilon\vec{E}$ and the crystal is optically isotropic and is equivalent to an amorphous body.

Group II. Crystals not belonging to group I in which two or more crystallographically-equivalent directions may be chosen in one plane. These are crystals of the trigonal, tetragonal and hexagonal systems, the plane containing the equivalent directions being perpendicular to the axis of three-fold, four-fold or six-fold symmetry. One dielectric principal axis must coincide with this distinguished direction, whilst for the other two one may choose any orthogonal line pair perpendicular to it. If the distinguished direction is taken as the z-axis one then has $\varepsilon_x = \varepsilon_y \neq \varepsilon_z$. Such crystals are said to be optically *uniaxial*.

Group III. Crystals in which no two crystallographically-equivalent directions may be chosen. These are the crystals belonging to the so-called orthorhombic, monoclinic and triclinic systems. Here $\varepsilon_x \neq \varepsilon_y \neq \varepsilon_z$ and the directions of the dielectric axes may or may not be determined by symmetry and may therefore be wavelength dependent. Crystals of this group are said to be optically *biaxial*.

That all crystals fall into these three types as regards their optical properties may be clearly seen by considering one of the associated ellipsoids, e.g. the ellipsoid of wave normals. Evidently the ellipsoid must be unchanged by the symmetry operations that leave the crystal structure unaltered. Now there are only two steps in the degeneration of an ellipsoid into a sphere: an ellipsoid has (a) all axes of unequal lengths or (b) two axes equal and one unequal (spheroid,

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i.e. ellipsoid of revolution), or (c) all axes equal (sphere); these correspond to the three groups which we just considered. The terms uniaxial and biaxial refer to the number of optic axes which the ellipsoid has, i.e. the number of diameters with the property that a plane section at right angles to them through the centre of the ellipsoid is a circle. A general ellipsoid has two such diameters (biaxial crystals), a spheroid has one such diameter (uniaxial crystals), and a sphere an infinite number (isotropic crystals) of diameters.

The following table shows a survey of the possible cases.

Crystal system	Ellipsoid of wave normals	Optical classification
Triclinic, Monoclinic, Orthorhombic	General Ellipsoid	Biaxial
Trigonal, Tetragonal, Hexagonal	Spheroid	Uniaxial
Cubic	Sphere	Isotropic

Light Propagation in Uniaxial Crystal:

For an optically uniaxial crystal ($\varepsilon_x = \varepsilon_y \neq \varepsilon_z$) with the optic axis in the z direction, we can write $v_x = v_y = v_o$ (say). We also write $v_z = v_e$.

Then Fresnel's Formula reduces to

$$\left(v_p{}^2 - v_o{}^2\right)\left[\left(s_x{}^2 + s_y{}^2\right)\left(v_p{}^2 - v_e{}^2\right) + s_z{}^2\left(v_p{}^2 - v_o{}^2\right)\right] = 0$$

Let θ denote the angle which the wave normal $\hat{s}$ makes with the z-axis, then $s_x{}^2 + s_y{}^2 = \sin^2\theta$ and $s_z{}^2 = \cos^2\theta$.

Therefore, we obtain $\left(v_p{}^2 - v_o{}^2\right)\left[\left(v_p{}^2 - v_e{}^2\right)\sin^2\theta + \left(v_p{}^2 - v_o{}^2\right)\cos^2\theta\right] = 0$.

The two roots of this equation (v'_p and v''_p say) are given by

$$v'_p{}^2 = v_o{}^2$$

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$$v''_p{}^2 = v_o{}^2\cos^2\theta + v_e{}^2\sin^2\theta$$

These equations show that the two shells of the normal surface are a sphere of radius $v'_p = v_o$ and a surface of revolution (of the fourth order), an ovaloid. Thus one of the two waves that correspond to any particular wave-normal direction is an *ordinary wave* (or o-wave), with a velocity (v_o) independent of the direction of propagation, the other an *extraordinary wave* (or e-wave) with velocity (v_e) depending on the angle between the direction of the wave normal and the optic axis. The two velocities are only equal when $\theta = 0$, i.e. when the wave normal is in the direction of the optic axis.

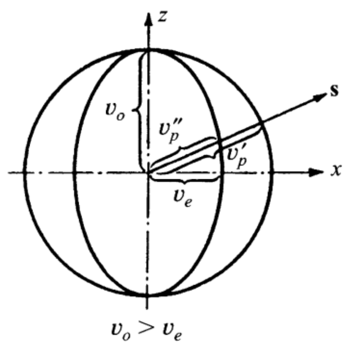

Fig. 1

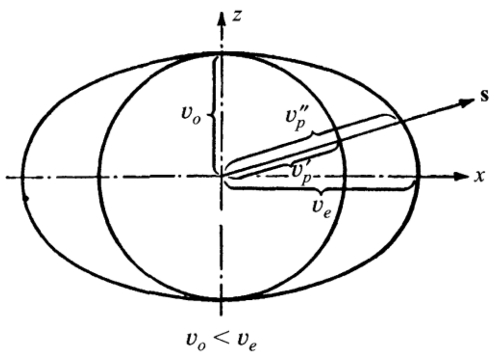

Fig. 2

When $v_o > v_e$ [see Fig. 1], the ordinary wave travels faster than the extraordinary wave (except for $\theta = 0$ when they are equal). Such a crystal is said to be a *positive uniaxial crystal* (e.g. quartz). If $v_o < v_e$ [see Fig. 2], the ordinary wave travels slower than the extraordinary wave, and we speak of a *negative uniaxial crystal* (e.g. felspar or calcite).

We also obtain two types of refractive indices. When the ordinary wave is moving, we obtain $n_o = \frac{c}{v_o}$, called *ordinary refractive index*. Similarly for the extraordinary wave we get *extraordinary refractive index* given by $n_e = \frac{c}{v_e}$.

<u>Double Refraction:</u>

In 1669 Erasmus Bartholinus, came upon a new and remarkable optical phenomenon in calcite, which he called double refraction, where double images

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were being formed when light was passing through calcite. If we send a narrow beam of natural light into a calcite crystal normal to a cleavage plane, it will split and emerge as two parallel beams. To see the same effect quite simply, we need only place a black dot on a piece of paper and then cover it with a calcite rhomb. The image will now consist of two gray dots (black where they overlap). Rotating the crystal will cause one of the dots to remain stationary while the other appears to move in a circle about it, following the motion of the crystal. The rays forming the fixed dot, which is the one invariably closer to the upper blunt corner, behave as if they had merely passed through a plate of glass. In accord with a suggestion made by Bartholinus, they are the ordinary rays, or o-rays. The rays coming from the other dot, which behave in such an unusual fashion, are the extraordinary rays, or e-rays. If the crystal is examined through an analyzer, it will be found that the ordinary and extraordinary images are linearly polarized (Fig. 3). Moreover, the two emerging polarization-states are orthogonal.

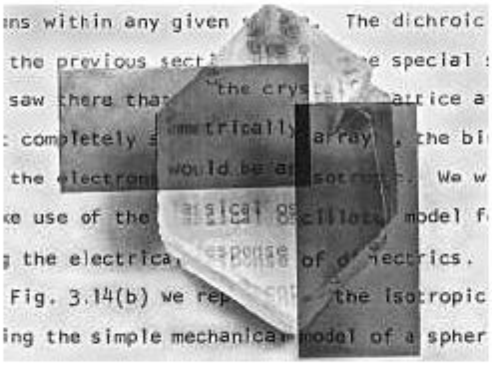

Fig. 3

Any number of planes can be drawn through the rhomb so as to contain the optic axis, and these are all called principal planes. More specifically, if the principal plane is also normal to a pair of opposite surfaces of the cleavage form, it slices the crystal across a principal section. Evidently, three of these pass through any one point; each is a parallelogram having angles of 109^0 and 71^0. Fig. 4 is a diagrammatic representation of an initially unpolarized beam traversing a principal section of a calcite rhomb. The filled-in circles and arrows drawn along the rays indicate that the o-ray has its electric-field vector normal to the principal section, and the field of the e-ray is parallel to the principal section.

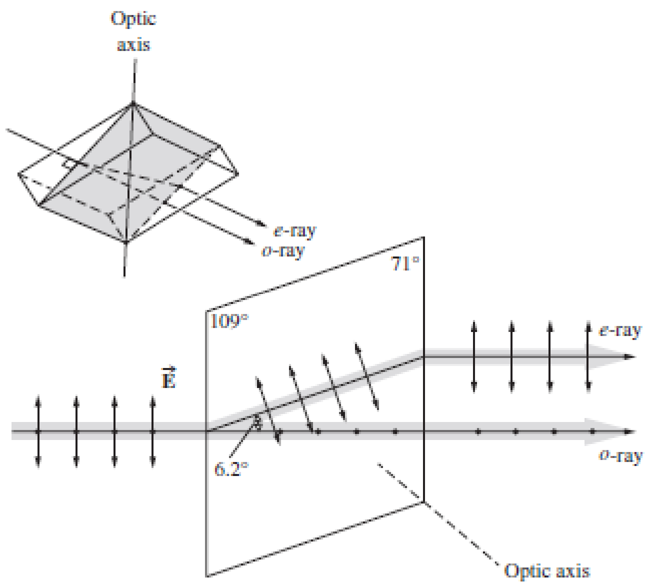

Fig. 4

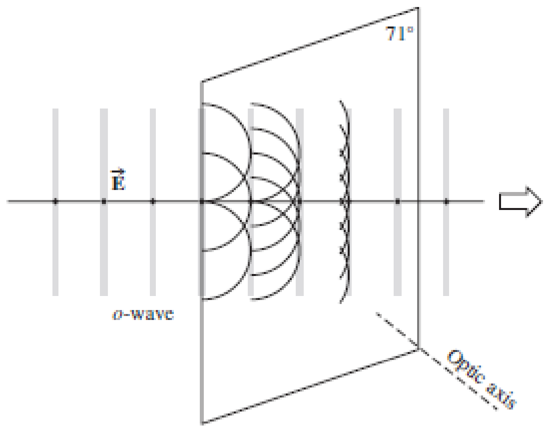

Fig. 5

To simplify matters a bit, let $\vec{E}$ in the incident plane wave be linearly polarized perpendicular to the optic axis, as shown in Fig. 5. The wave strikes the surface of the crystal, thereupon driving electrons into oscillation, and they in turn reradiate secondary wavelets. The wavelets superimpose and recombine to form

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the refracted wave, and the process is repeated over and over again until the wave emerges from the crystal. This represents a cogent physical argument for applying the ideas of scattering via Huygens's Principle. Huygens himself, though without benefit of electromagnetic theory, used his construction to explain many aspects of double refraction in calcite as long ago as 1690.

Nicol Prism:

One of the most commonly used instruments for the production of linearly polarized light is the Nicol prism. It consists of a natural rhomb of calcite (uniaxial crystal) which is cut into two equal parts along a diagonal plane (represented by AC in Fig. 6), and with the two parts cemented together with Canada balsam. The rhomb is about three times as long as it is wide, with the angles at B and D of its principal section equal to 71^0; the end faces AD and BC are ground off to reduce this angle to 68^0.

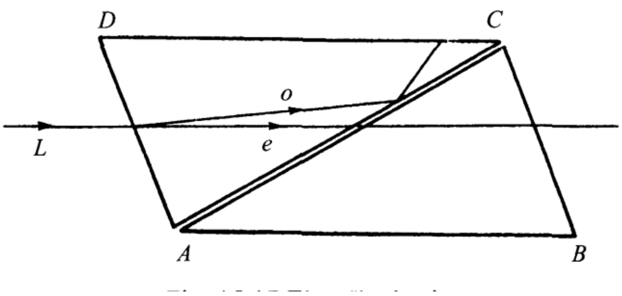

Fig. 6

A ray of light incident in the direction L parallel to the long edge is split into an ordinary ray and an extraordinary ray. For the first ray the Canada balsam is of lower, for the second of higher, optical density (calcite crystal: $n_o = 1.66$, $n_e = 1.49$; Canada balsam $n = 1.53$), and it may easily be verified that on the Canada balsam interfaces conditions for total reflection are satisfied with respect to the ordinary ray; this ray is totally reflected towards the face DC which is blackened and so absorbs it. The extraordinary ray passes through the prism with practically no lateral displacement and is linearly polarized with its $\vec{D}$ vector in the principal section. Thus the Nicol prism produces linearly polarized light, whose direction of vibration is known.

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If the incident ray is inclined to the edge of the rhomb the Nicol prism will still act as a polarizer, provided that for the ordinary ray the angle of incidence on the Canada balsam is not smaller than the critical angle; this limits the angle of the cone of incident rays in air for which the prism is effective to about 30^0.

Half-Wave Plate:

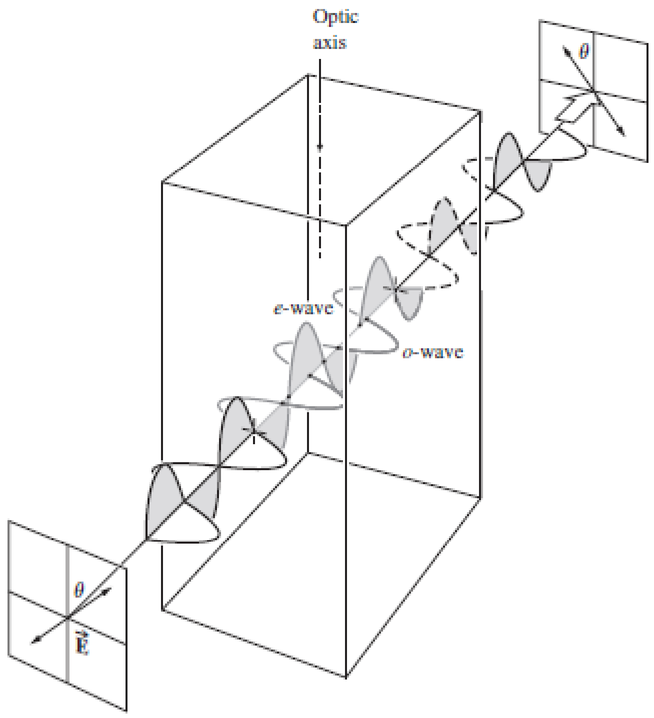

Fig. 7

A retardation plate that introduces a relative phase difference of π radians, or 180^0 between the ordinary (o-wave) and extraordinary (e-wave) waves is known as a *half-wave plate* or *half-wave retarder*. Suppose that the plane of vibration of an incoming beam of linear light makes some arbitrary angle θ with the fast axis, as shown in Fig. 7. In a negative material the e-wave will have a higher speed (same n) and a longer wavelength than the o-wave. When the waves emerge from the plate, there will be a relative phase shift of $\frac{\lambda}{2}$ (that is, π radians), with the effect that $\vec{E}$ will have rotated through 2θ (Fig. 8). In fact,

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half-wave retarders are sometimes called polarization rotators for just that reason. It should be evident that a half-wave plate will similarly flip elliptical light. In addition, it will invert the handedness of circular or elliptical light, changing right to left and vice versa. A half-wave plate shifts the polarization states halfway around.

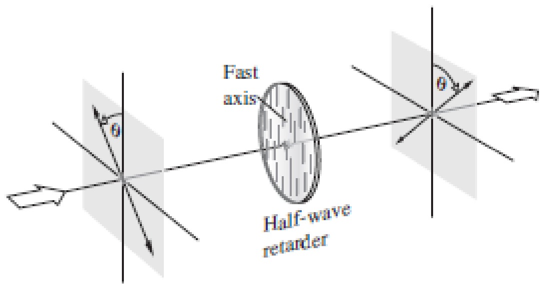

Fig. 8

As the e-wave and o-wave progress through any retardation plate, their relative phase difference $\Delta\phi$ increases, and the state of polarization of the wave therefore gradually changes from one point in the plate to the next. Evidently, if the thickness of the material is such that

$$d(|n_o - n_e|) = (2m+1)\frac{\lambda_0}{2}$$

where $m = 0, 1, 2, \ldots,$ it will function as a half-wave plate.

Quarter-Wave Plate:

The quarter-wave plate is an optical element that introduces a relative phase shift of $\Delta\phi = \frac{\pi}{2}$ between the constituent orthogonal o- and e-components of a wave. It follows once again that a phase shift of 90^0 will convert linear to elliptical light and vice versa. It should be apparent that linear light incident parallel to either principal axis will be unaffected by any sort of retardation plate. Here we can't have a relative phase difference without having two components. With incident natural light, the two constituent polarization-states are incoherent; that is, their relative phase difference changes randomly and rapidly. The introduction of an additional constant phase shift by any form of retarder will still result in a random phase difference and thus have no noticeable effect.

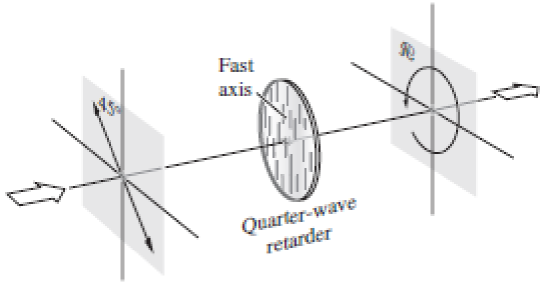

Fig. 9

When linear light at 45^0 to either principal axis is incident on a quarter-wave plate, its o- and e-components have equal amplitudes. Under these special circumstances, a 90^0 phase shift converts the wave into circular light (Fig. 9). Similarly, an incoming circular beam will emerge linearly polarized. Whenever linear light is converted to either elliptical or circular light by a quarter-wave plate, the resulting handedness corresponds to the same direction it would take to rotate the initial linear light into alignment with the slow axis, through the smallest angle.

Quarter-wave plates are also usually made of quartz, mica, or organic polymeric plastic. In any case, the thickness of the birefringent material must satisfy the expression

$$d(|n_o - n_e|) = (4m + 1)\frac{\lambda_0}{4}$$

where $m = 0, 1, 2, \ldots$.

Babinet Compensator:

A compensator is an optical device that is capable of impressing a controllable retardance on a wave. Unlike a wave plate where phase difference $\Delta\phi$ is fixed, the relative phase difference arising from a compensator can be varied continuously. The Babinet compensator, depicted in Fig. 10, consists of two independent calcite, or more commonly quartz, wedges whose optic axes are

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indicated by the lines and dots in the figure. A ray passing vertically downward through the device at some arbitrary point will traverse a thickness of d_1 in the upper wedge and d_2 in the lower one. The relative phase difference imparted to the wave by the first crystal is $\frac{2\pi}{\lambda_0}d_1(|n_o - n_e|)$, and that of the second crystal is $-\frac{2\pi}{\lambda_0}d_2(|n_o - n_e|)$.

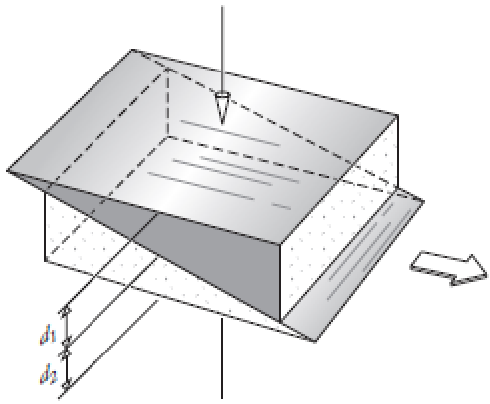

Fig. 10

The compensator is thin (the wedge angle is typically about 2.5^0), and thus the separation of the rays is negligible. The total phase difference or retardance, is then given by

$$\Delta\phi = \frac{2\pi}{\lambda_0}(d_1 - d_2)(|n_o - n_e|)$$

If the compensator is made of calcite, the e-wave leads the o-wave in the upper wedge, and therefore if $d_1 > d_2$, $\Delta\phi$ corresponds to the total angle by which the e-component leads the o-component. The converse is true for a quartz compensator. In other words, if $d_1 > d_2$, $\Delta\phi$ is the angle by which the o-wave leads the e-wave.

At the centre, where $d_1 = d_2$, the effect of one wedge is exactly cancelled by the other, and $\Delta\phi = 0$ for all wavelengths. The retardation will vary from point to point over the surface, being constant in narrow regions running the width of the compensator along which the wedge thicknesses are themselves constant. If light enters by way of a slit parallel to one of these regions and if we then move

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either wedge horizontally with a micrometer screw, we can get any desired $\Delta\phi$ to emerge.

When the Babinet is positioned at 45^0 between crossed polarizers, a series of parallel, equally spaced, dark extinction fringes will appear across the width of the compensator. These mark the positions where the device acts as if it was a full-wave plate. In white light the fringes will be coloured, with the exception of the black central band ($\Delta\phi = 0$). The retardance of an unknown plate can be found by placing it on the compensator and examining the fringe shift it produces. Because the fringes are narrow and difficult to read electronically, the Babinet has become less popular than it once was.

This concludes part 2 of this e-report.

The discussion will be continuing in the part 3 of this e-report.

Reference(s):

Principles of Optics, Max Born & Emil Wolf, Cambridge University Press

Optics, Eugene Hecht, Pearson Education

(All the figures have been collected from the above mentioned references)