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Topic:

Complex Analysis (3): Laurent's Theorem, Laurent's Series, Complex Integration Round Unit Circle of the Type, Evaluation of Complex Integration in Limit Range from $-\infty$ to $+\infty$ and 0 to ∞

COMPLEX ANALYSIS (3)

Laurent's Theorem:

If the function $f(z)$is analytic on C_1 and C_2, and the annular region R bounded the two concentric circles C_1 and C_2 of radii $r_1$and r_2 ($r_2<r_1$) and with the centre a, then for all z in R:

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + .. + \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + ...$$

Where, $a_n = \frac{1}{2\pi i}\oint_{C1}\frac{f(w)}{(w-a)^{n+1}}dw$

And $b_n = \frac{1}{2\pi i}\oint_{C2}\frac{f(w)}{(w-a)^{-n+1}}dw$

Problem-1:

Find the Laurent's Series Expansion of $f(z) = \frac{z}{(z-1)(z-2)}$ valid for $|z-1| > 1$

Answer: $f(z) = \frac{z}{(z-1)(z-2)}$

$$f(z) = \frac{z}{(z-1)(z-2)} = \frac{-1}{(z-1)} + \frac{2}{(z-2)} = \frac{-1}{(z-1)} + \frac{2}{(z-1)-1}$$

$$= \frac{-1}{(z-1)} + \frac{2}{(z-1)}\frac{1}{1-\frac{1}{(z-1)}} = \frac{-1}{(z-1)} + \frac{2}{(z-1)}\left(1-\frac{1}{(z-1)}\right)^{-1}$$

$$= \frac{-1}{(z-1)} + \frac{2}{(z-1)}\left(1 + \frac{1}{(z-1)} + \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} + ...\right)$$

$$= \frac{-1}{(z-1)} + \frac{2}{(z-1)} + \frac{2}{(z-1)^2} + \frac{2}{(z-1)^3} + \frac{2}{(z-1)^4} + ...$$

$$= \frac{1}{(z-1)} + \frac{2}{(z-1)^2} + \frac{2}{(z-1)^3} + \frac{2}{(z-1)^4} + ...$$

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Problem-2:

Evaluate the following integral

$$\int_0^{2\pi} \frac{Cos\ 2\theta}{5+4\ Cos\ \theta} d\theta$$ **by using Contour integration**

I = $\int_0^{2\pi} \frac{Cos\ 2\theta}{5+4\ Cos\ \theta} d\theta$

= Real part of $\int_0^{2\pi} \frac{Cos\ 2\theta + i\ Sin\ 2\theta}{5+4\ Cos\ \theta} d\theta$

Taking, $e^{i\theta} = z$

$e^{i\theta}\ d\theta = z\ dz$

$d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$

= Real part of $\int_0^{2\pi} \frac{e^{2i\theta}}{5+2(e^{i\theta}+e^{-i\theta})} d\theta$

=Real part of $\int_C \frac{z^2}{5+2(z+\frac{1}{z})} \frac{dz}{iz}$

=Real part of $\int_C \frac{z^2}{5z+2z^2+2} \frac{dz}{i}$

=Real part of $\int_C \frac{-iz^2}{5z+2z^2+2} dz$

=Real part of $\int_C \frac{-iz^2}{(2z+1)(z+2)} dz$

Poles are determined by putting denominator equal to zero

Now, $(2z+1)(z+2) = 0$

Therefore, $z = -\frac{1}{2}, -2$

The only simple pole at $z = -\frac{1}{2}$ is inside the contour,

The residue of $f(z)$ (at $z = -\frac{1}{2}$) $= \lim_{z \to -\frac{1}{2}} \left(z+\frac{1}{2}\right) f(z)$

Therefore, $= \lim_{z \to -\frac{1}{2}} \left(z+\frac{1}{2}\right) \frac{-iz^2}{(2z+1)(z+2)}$

Therefore, $= \underset{z\to -\frac{1}{2}}{Lim} \frac{-iz^2}{2(z+2)} = \frac{-i}{12}$

By, the Cauchy's integral theorem we know that the

$$\int_C f(z)dz = 2\pi i \,(sum\ of\ the\ residues\ at\ the\ poles\ within\ C)$$

$= 2\pi i$ x $(\frac{-i}{12}) = \frac{\pi}{6}$, which is real

Therefore, $\int_0^{2\pi} \frac{Cos\ 2\theta}{5+4\ Cos\ \theta} d\theta = \frac{\pi}{6}$

Problem-3:

Evaluate the following integral

$$\int_{-\infty}^{\infty} \frac{xSin\pi x}{5 + x^2 + 2x} dx$$ **by using complex variable**

We have now, $\int_{-\infty}^{\infty} \frac{x\ Sin\ \pi x}{5+x^2+2x} dx$

Let us consider, $\int_c \frac{z\ Sin\ \pi z}{z^2+2z+5} dz$

The pole can be determined by putting the denominator equal to zero.

So, $z^2 + 2z + 5 = 0$

Hence, $z = \frac{-2\pm\sqrt{4-20}}{2} = -1 \pm 2i$

Out of two poles, only $z = -1 + 2i$ is inside the contour.

Residue at $z = -1 + 2i$

The residue of $f(z)$ (at $z = -1 + 2i$) $= \underset{z\to -1+2i}{Lim} (z + 1 - 2i) \frac{z\ Sin\ \pi z}{z^2+2z+5}$

$= \underset{z\to -1+2i}{Lim} \frac{z\ Sin\ \pi z}{(z+1-2i)(z+1+2i)}$

$= \underset{z\to -1+2i}{Lim} (z + 1 - 2i) \frac{z\ Sin\ \pi z}{(z+1+2i)}$

$= (-1 + 2i) \frac{Sin\ \pi\ (-1+2i)}{((-1+2i)+1+2i)}$

$=(-1+2i)\frac{Sin\ \pi\,(-1+2i)}{4i}$

Therefore, $\int_{-R}^{R}\frac{z\ Sin\ \pi z}{z^2+2z+5}dz\ =2\pi i\ (Residue)$

$=2\pi i(-1+2i)\frac{Sin\ \pi\,(-1+2i)}{4i}$

$=\frac{\pi}{2}(-1+2i)(-Sin\ \ 2\pi i)$

$=\frac{\pi}{2}(1-2i)(Sin\ \ 2\pi i)$

$=\frac{\pi}{2}(1-2i)iSinh\ \ 2\pi$

$=\frac{\pi}{2}(i+2)Sinh\ \ 2\pi$

(Taking Real Parts)

Hence, $\int_{-\infty}^{\infty}\frac{x\ Sin\ \pi x}{5+x^2+2x}dx\ =\ Sinh\ \ 2\pi$

Problem-4:

Evaluate the following integral

$$\int_0^{\infty}\frac{Cos\ 3x}{(x^2+1)(x^2+4)}dx$$ **by using complex variable**

Let, $f(z)\ =\frac{e^{3iz}}{(z^2+1)(z^2+4)}$

The pole can be determined by putting the denominator equal to zero.

So, $(z^2+1)(z^2+4)=0$

Hence, $(z^2+1)=0\ or, z=\ \pm i$

And $(z^2+4)=0\ or, z=\ \pm 2i$

Let C be the closed contour consisting of the upper half C_R of a large circle $|z|=R$ and the real axis from $-R$ to $+R$. The poles at $z=i$ and $z=2i$ lie within the contour.

The residue of $f(z)$ (at $z=i$) $=\lim_{z\to i}(z-i)\frac{e^{3iz}}{(z^2+1)(z^2+4)}$

$=\lim_{z\to i}\frac{e^{3iz}}{(z+i)(z^2+4)}=\frac{e^{-3}}{6i}$

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The residue of $f(z)$ (at $z = 2i$) $= \lim_{z\to 2i}(z-2i)\frac{e^{3iz}}{(z^2+1)(z^2+4)}$

$= \lim_{z\to 2i}\frac{e^{3iz}}{(z+2i)(z^2+1)} = \frac{e^{-6}}{-12i}$

By theorem of Residue, $\int_c f(z)dz = 2\pi i\,(Sum\ of\ the\ Residues)$

$$\int_{-R}^{R}\frac{e^{3iz}}{(z+2i)(z^2+1)}dz + \int_{CR}\frac{e^{3iz}}{(z+2i)(z^2+1)}dz = 2\pi i\left[\frac{e^{-3}}{6i}+\frac{e^{-6}}{-12i}\right]$$

Here, $\int_{CR}\frac{e^{3iz}}{(z+2i)(z^2+1)}dz = 0$

$\int_0^{\infty}\frac{Cos\ 3x}{(x^2+1)(x^2+4)}dx$ = Real Part of $\frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{3ix}}{(x^2+1)(x^2+4)}dx$

= Real Part of $\frac{\pi}{2}\left(\frac{e^{-3}}{3}+\frac{e^{-6}}{6}\right)$

Hence, the given integral = $\frac{\pi}{2}\left(\frac{e^{-3}}{3}+\frac{e^{-6}}{6}\right)$

Frequently Asked Questions/Numerical:

For more problems in this section of Complex Analysis, students can solve the problems of Mathematical Physics: Author-H.K. Dass& Rama Verma, Published by S. Chand (2018 Ed.) and Principles of Mathematical Physics, Author- S. P. Kuila, Published by NCBA (2018 Ed.).

References:

(i) *Mathematical Physics: Author-H.K. Dass& Rama Verma,Published by S. Chand(2018 Ed.).*
(ii) *Principles of Mathematical Physics, Author- S. P. Kuila, Published by NCBA (2018 Ed.).*

Link to Audio visual Lectures (e-Lectures) on this topic given by Distinguish Professors of Indian &Foreign Universities:

(i) https://nptel.ac.in/courses/112108285/
(ii) https://nptel.ac.in/courses/115106086/
(iii) http://www.infocobuild.com/education/audio-video-courses/physics/selected-topics-in-mathematical-physics-iit-madras.html
(iv) https://www.youtube.com/watch?v=BPmVO4OT_Hk
(v) https://www.youtube.com/watch?v=SwrtCLbDQu8
(vi) https://www.youtube.com/watch?v=nDD16hiutdc
(vii) https://www.youtube.com/watch?v=Iiom5V5faxc
(viii) https://www.youtube.com/watch?v=b5VUnapu-qs
(ix) https://www.youtube.com/watch?v=Jkv-55ndVYY
(x) https://www.youtube.com/watch?v=kik1m6fjutA