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Topic:

Complex Analysis (2): Residue (Definition), Residue at Infinity, Formula of Finding Residue, Taylor's Series Expansion

COMPLEX ANALYSIS (2)

Residue:

Let $z = a$ be a pole of order m of a function $f(z)$ and C_1 of radius r with centre at z=a which does not contain any other singularities except at $z = a$ then $f(z)$ is analytic within the annulus $r < |z - a| < R$ can be expanded within the annulus. The Lauren's series:

$f(z) = \sum_0^\infty a_n (z-a)^n + \sum_1^\infty b_n (z-a)^{-n}$

Where, $a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz$

And $b_n = \frac{1}{2\pi i} \oint_{C1} \frac{f(z)}{(z-a)^{-n+1}} dz$

And $|z - a| = r$ being the circle C_1

For more particular, $b_1 = \frac{1}{2\pi i} \oint_{C1} f(z) dz$

The coefficient b_1 is called the residue of $f(z)$ at the pole $z = a$

It is denoted by symbol Res.($z = a$) = b_1

Residue at infinity:

Residue of $f(z)$ $at\ z = \infty$ is defined as $-\frac{1}{2\pi i} \oint_C f(z) dz$ where the integration is taken round C in anti-clock direction.

Where C is a large circle containing all finite singularities of $f(z)$

Formula of Finding Residue:

(a) Residue at Simple Pole

(i) If $f(z)$ has a simple pole at $z = a$, then

Res $f(a) = \lim_{z \to a} (z-a) f(z)$

(ii) If $f(z)$ is the form of $f(z) = \frac{\varphi(z)}{\psi(z)}$ where, $\psi(a) = 0$, but $\psi(a) \neq 0$

Then,Residue (at $z = a$) $= \frac{\varphi(a)}{\psi'(a)}$

(b) Residue at a Pole of order n:

If $f(z)$ has a pole of order n at $z = a$, then

Then,Residue (at $z = a$) $= \frac{1}{(n-1)!}\{\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]\}$

(c) Residue at a pole $z = a$ of any order (simple or of order m):

Res $f(a)$= Coefficient of $\frac{1}{t}$

(d) Residue of $f(z)$ at $z = \infty$

$$\lim_{z \to \infty} \{-z f(z)\}$$

Residue of $f(z)$ $at\ z = \infty$ is defined as $-\frac{1}{2\pi i}\oint_C f(z)dz$

Problem-1:

Determine the pole and residue at the pole of the function $f(z) = \frac{z}{z-1}$

The pole of $f(z)$ are given by putting the denominator equal to zero

Therefore, $z - 1 = 0$

Hence $z = 1$

The residue is calculated by formula:Res $f(a) = \lim_{z \to a} (z-a) f(z)$

The residue of $f(z)$ (at $z = 1$) $= \lim_{z \to 1} (z-1)\frac{z}{z-1}$

Or, $\lim_{z \to 1} z = 1$

Hence, $f(z)$ has a simple pole of at $z = 1$ and residue at the pole is 1.

Problem-2:

Determine the poles and residue at the simple pole of the function $f(z) = \frac{z^2}{(z-1)^2(z+2)}$

Answer: The pole of $f(z)$ are given by putting the denominator equal to zero

Therefore, $(z-1)^2(z+2) = 0$

Hence $z = 1,1,-2$

The function $f(z)$ has simple pole at $z = -2$ and at $z = 1$ pole of second order

The residue of $f(z)$ (at $z = -2$) $= \frac{Lim}{z \to -2}(z+2)\frac{z^2}{(z-1)^2(z+2)}$

Therefore, $= \frac{Lim}{z \to -2}\frac{z^2}{(z-1)^2}$

$= \frac{Lim}{z \to -2}\frac{(-2)^2}{(-2-1)^2} = \frac{4}{9}$

Problem-3:

Determine the poles and residue of the function $f(z) = \frac{z}{Sin\ z}$

Answer: The pole of $f(z)$ are given by putting the denominator equal to zero

Therefore, $Sin\ z = 0$

Hence $z = n\pi$

The function $f(z)$ has simple pole at $z = -2$ and at $z = 1$ pole of second order

The residue of $f(z)$ (at $z = n\pi$) $= (\frac{z}{Cos\ z})_{z=n\pi}$

Therefore, $= \frac{n\pi}{Cos\ n\pi}$

$= \frac{n\pi}{(-1)^n}$

Hence, the residue of the given function at pole $z = n\pi$ is $\frac{n\pi}{(-1)^n}$

Problem-4:

Find the sum of the Residuesof the function $f(z) = \frac{Sin\ z}{z\ Cos\ z}$ **at its pole inside the circle**

of $|z| = 2$

Answer: The pole of $f(z)$ are given by putting the denominator equal to zero

Therefore, $z\, Cos\, z = 0$

Hence $z = 0, \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \ldots \ldots \ldots$

Of the poles only $z = 0$ and at $z = \pm \frac{\pi}{2}$ lie inside the circle of $|z| = 2$

The residue of $f(z)$ (at $z = 0$) $= \frac{Lim}{z \to 0} \frac{Sin\, z}{Cos\, z} = 0$

The residue of $f(z)$ (at $z = \frac{\pi}{2}$) $= \frac{Lim}{z \to 0} \frac{\left(z - \frac{\pi}{2}\right) Sin\, z}{z\, Cos\, z} = -\frac{2}{\pi}$

Hence, the sum of the residues=0 $-\frac{2}{\pi} + \frac{2}{\pi} = 0$

Residue Theorem:

If $f(z)$ is analytic in a closed curve C, except at a finite number of poles within C, then

the integral $\int_C f(z)dz = 2\pi i$ $(sum\ of\ the\ residues\ at\ the\ poles\ within\ C)$

Problem-5:

Evaluate the following integral using residue theorem

$$\int_C \frac{(1+z)}{z(z-2)} dz, \text{ where C is the circle } |z| = 1$$

Poles of the integrand are given by putting the denominator equal to zero.

So, $z(z-2) = 0$

Therefore, $z = 0,2$

The integrand is analytic on $|z| = 1$ and all points inside, except$z = 0$, as a pole at $z = 0$ is inside the circle $|z| = 1$

Hence by residue theorem

$\int_C \frac{(1+z)}{z(z-2)} \mathrm{dz} = 2\pi i \,[Res\, f(0)]$

Hence, $Res\, f(0) = \lim_{z\to 0} z\, \frac{(1+z)}{z(2-z)}$

Therefore, $\lim_{z\to 0} \frac{(1+z)}{(2-z)} = \frac{1}{2}$

$\int_C \frac{(1+z)}{z(z-2)} \mathrm{dz} = 2\pi i \left[\frac{1}{2}\right] = \pi i$

Problem-6:

Evaluate the following integral using residue theorem

$$\int_C \frac{(4-3z)}{z(z-1)(z-2)} \mathrm{dz}, \text{where C is the circle } |z| = \frac{3}{2}$$

Poles of the integrand are given by putting the denominator equal to zero.

So, $z(z-1)(z-2) = 0$

Therefore, $z = 0,1,2$

The has a poles at $z = 0, 1, 2$of which the given circle encloses the poles at $z = 0, 1$is inside the circle $|z| = \frac{3}{2}$

The residue of $f(z)$ at the simple pole (at $z = 0$) $= \lim_{z\to 0} z\, \frac{(4-3z)}{z(z-1)(z-2)}$

Hence, $= \lim_{z\to 0} \frac{(4-3z)}{(z-1)(z-2)}$

Hence, $= \frac{(4-0)}{(0-1)(0-2)} = 2$

The residue of $f(z)$ at the simple pole (at $z = 1$) $= \lim_{z\to 1} (z-1)\, \frac{(4-3z)}{z(z-1)(z-2)}$

Hence, $= \lim_{z\to 1} \frac{(4-3z)}{z(z-2)}$

Hence, $= \frac{(4-3)}{(1)(1-2)} = -1$

Hence by residue theorem

$\int_C \frac{(4-3z)}{z(z-1)(z-2)} \mathrm{dz} = 2\pi i\,(2-1) = 2\pi i$

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Problem-7:

Find the first four terms of the Taylor's series expansion of the complex variable function $f(z) = \frac{z+1}{(z-3)(z-4)}$ about $(z) = 2$.Find the region of convergence

Answer: $f(z) = \frac{z+1}{(z-3)(z-4)}$

If centre of circle is at $z = 0$, then the distance of the singularities$z = 3$ and $z = 4$ from the centre are 1 and 2. Hence, if a circle is drawn with centre $z = 2$ and 1, then within the circle $|z - 2| = 1$, the given function $f(z)$ is analytic, hence it can be expanded in a Taylor's series within the circle $|z - 2| = 1$, the circle is convergence.

$f(z) = \frac{z+1}{(z-3)(z-4)} = \frac{-4}{(z-3)} + \frac{5}{(z-4)} = \frac{-4}{(z-2)-1} + \frac{5}{(z-2)-2} = 4[1-(z-2)]^{-1} - \frac{5}{2}[1-\frac{(z-2)}{2}]^{-1}$

$= 4[1+(z-2)+(z-2)^2+(z-1)^3+\cdots] - \frac{5}{2}[1-\frac{(z-2)}{2}+\frac{(z-2)^2}{4}+\frac{(z-2)^3}{8}+\cdots]$

$= \left(4-\frac{5}{2}\right)+\left(4-\frac{5}{4}\right)(z-2)+\left(4-\frac{5}{8}\right)(z-2)^2+\left(4-\frac{5}{16}\right)(z-2)^3+\cdots$

$= \frac{3}{2}+\frac{11}{4}(z-2)+\frac{27}{8}(z-2)^2+\frac{59}{16}(z-2)^3+\cdots$

Frequently Asked Questions/Numerical:

For more problems in this section of Complex Analysis, students can solve the problems of Mathematical Physics: Author-H.K. Dass& Rama Verma, Published by S. Chand (2018 Ed.) and Principles of Mathematical Physics, Author- S. P. Kuila, Published by NCBA (2018 Ed.).

References:

(i) *Mathematical Physics: Author-H.K. Dass& Rama Verma,Published by S. Chand(2018 Ed.).*

(ii) *Principles of Mathematical Physics, Author- S. P. Kuila, Published by NCBA (2018 Ed.).*

Link to Audio visual Lectures (e-Lectures) given by Distinguish Professors of Indian &Foreign Universities:

(i) https://www.youtube.com/watch?v=n8TIspt2_-g
(ii) https://www.youtube.com/watch?v=23HdJmVpsao
(iii) https://nptel.ac.in/courses/111103070/
(iv) https://www.youtube.com/watch?v=z03usEpsHRU
(v) https://www.youtube.com/watch?v=_M6-hkirtn4
(vi) https://www.youtube.com/watch?v=y-kHuYvLoOI
(vii) https://www.youtube.com/watch?v=Mpmlk1H1aQo

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