Dr. Avradip Pradhan,
Assistant Professor,
Department of Physics,
Narajole Raj College, Narajole.

C13T (Electromagnetic Theory)

Topic – EM Wave in Bounded Media (Part – 2)

We have already discussed part 1 of this e-report.

Now let us continue part 2 of it.

Boundary Conditions for Oblique Incidence at the Interface between two Media:

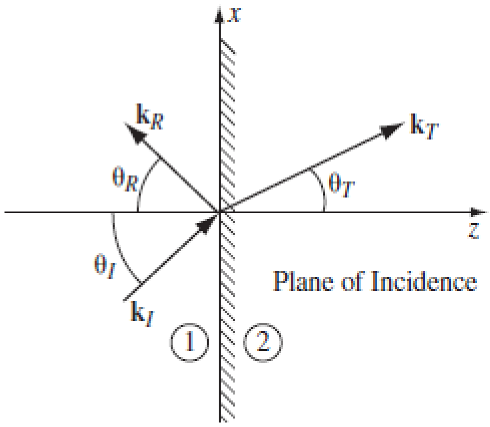

Fig. 1

We have already written the electric and magnetic field values for the incident, reflected and transmitted (or refracted) waves as (shown in Fig. 1)

$$\vec{E}_I(\vec{r},t) = \vec{E}_{0I}e^{i(\vec{k}_I.\vec{r}-\omega t)} \text{ and } \vec{B}_I(\vec{r},t) = \frac{1}{v_1}(\hat{k}_I \times \vec{E}_I)$$

$$\vec{E}_R(\vec{r},t) = \vec{E}_{0R}e^{i(\vec{k}_R.\vec{r}-\omega t)} \text{ and } \vec{B}_R(\vec{r},t) = \frac{1}{v_1}(\hat{k}_R \times \vec{E}_R)$$

$$\vec{E}_T(\vec{r},t) = \vec{E}_{0T}e^{i(\vec{k}_T.\vec{r}-\omega t)} \text{ and } \vec{B}_T(\vec{r},t) = \frac{1}{v_2}(\hat{k}_T \times \vec{E}_T)$$

We have also got the boundary conditions to be obeyed at the interface between two media

$$\varepsilon_1 E_1^{\perp} = \varepsilon_2 E_2^{\perp} \qquad \vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}$$

$$B_1^{\perp} = B_2^{\perp} \qquad \frac{1}{\mu_1}\vec{B}_1^{\parallel} = \frac{1}{\mu_2}\vec{B}_2^{\parallel}$$

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Now using these boundary conditions at the interface ($z = 0$) we obtain

$$\varepsilon_1(\vec{E}_{0I} + \vec{E}_{0R})_z = \varepsilon_2(\vec{E}_{0T})_z$$

$$(\vec{B}_{0I} + \vec{B}_{0R})_z = (\vec{B}_{0T})_z$$

$$(\vec{E}_{0I} + \vec{E}_{0R})_{x,y} = (\vec{E}_{0T})_{x,y}$$

$$\frac{1}{\mu_1}(\vec{B}_{0I} + \vec{B}_{0R})_{x,y} = \frac{1}{\mu_2}(\vec{B}_{0T})_{x,y}$$

Here $\vec{B}_0 = \frac{1}{v}(\hat{k} \times \vec{E}_0)$ in each case. The last two represent pairs of equations, one for the x −component and one for the y −component.

Now there are two types of polarization which can be assumed. One polarization (direction of the electric field) is *parallel* to the plane of incidence ($x - z$ plane), and the other one is *perpendicular or normal* to the plane of incidence.

<u>Polarization Parallel to the Plane of Incidence:</u>

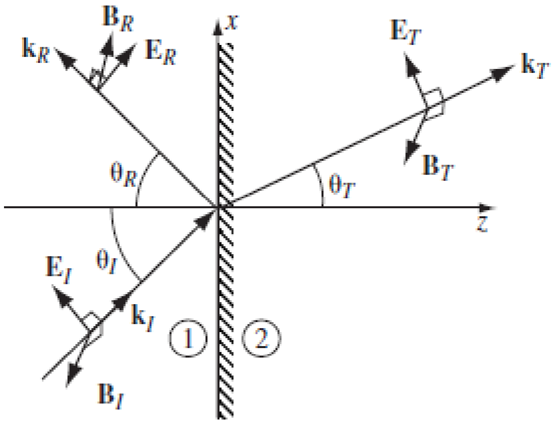

Fig. 2

In this case we also know that the reflected and transmitted waves also get polarized in this plane (as shown in Fig. 2). So, we obtain from the boundary condition

$$\varepsilon_1(-E_{0I}\sin\theta_I + E_{0R}\sin\theta_R) = \varepsilon_2(-E_{0T}\sin\theta_T)$$

Dr. Avradip Pradhan,
Assistant Professor,
Department of Physics,
Narajole Raj College, Narajole.

$$0 = 0 \text{ (trivial)}$$

$$E_{0I}\cos\theta_I + E_{0R}\cos\theta_R = E_{0T}\cos\theta_T$$

$$\frac{1}{\mu_1 v_1}(E_{0I} - E_{0R}) = \frac{1}{\mu_2 v_2}E_{0T}$$

By a little bit of algebra, it can be shown that both equations 1 and 4 are the same, which can be reduced to $E_{0I} - E_{0R} = \frac{\mu_1 v_1}{\mu_2 v_2}E_{0T} = \beta E_{0T}$, where $\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}$.
Now from equation 3, we obtain $(E_{0I} + E_{0R})\cos\theta_I = E_{0T}\cos\theta_T$. Therefore, we get $E_{0I} + E_{0R} = \alpha E_{0T}$, where $\alpha = \frac{\cos\theta_T}{\cos\theta_I}$.

Fresnel's Equations for Parallel Polarization. Solving the previous two equations we finally get

$$E_{0R} = \frac{\alpha - \beta}{\alpha + \beta}E_{0I}$$

$$E_{0T} = \frac{2}{\alpha + \beta}E_{0I}$$

These two equations are known as *Fresnel's Equations for Parallel Polarization*.

Reflection and Transmission Coefficients. Two important parameters known as reflection coefficient (R) and transmission coefficient (T) can be defined as

$$R = \frac{I_R}{I_I} \text{ and } T = \frac{I_T}{I_I}$$

Here I_I, I_R and I_T are known as the intensity of the incident, reflected and transmitted waves. Usually the intensity (average power per unit area) is defined as $I = \frac{1}{2}\varepsilon v {E_0}^2$.

In this case, the incident intensity will be $I_I = \frac{1}{2}\varepsilon_1 v_1 {E_{0I}}^2 \cos\theta_I$. The $\cos\theta_I$ term will come since the incident energy is falling at an angle θ_I with the normal.

Dr. Avradip Pradhan,
Assistant Professor,
Department of Physics,
Narajole Raj College, Narajole.

Similarly the reflected intensity will be $I_R = \frac{1}{2}\varepsilon_1 v_1 {E_{0R}}^2 \cos\theta_R$ and transmitted energy will be $I_T = \frac{1}{2}\varepsilon_2 v_2 {E_{0T}}^2 \cos\theta_T$.

Therefore, we get $R = \frac{I_R}{I_I} = \frac{{E_{0R}}^2}{{E_{0I}}^2} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2$ and $T = \frac{I_T}{I_I} = \frac{\varepsilon_2 v_2 {E_{0T}}^2 \cos\theta_T}{\varepsilon_1 v_1 {E_{0I}}^2 \cos\theta_I} = \alpha\beta\left(\frac{2}{\alpha+\beta}\right)^2 = \frac{4\alpha\beta}{(\alpha+\beta)^2}$.

It is important to note that $R + T = 1$, which establishes the *conservation of energy*, which is the energy per unit time reaching a particular patch of area on the surface is equal to the energy per unit time leaving the patch.

Polarization Perpendicular to the Plane of Incidence:

In this case we see that the reflected and transmitted waves also get polarized perpendicular to the plane of incidence ($x - z$ plane). So, we obtain from the boundary conditions

$$0 = 0 \text{ (trivial)}$$

$$\frac{1}{v_1}(E_{0I}\sin\theta_I + E_{0R}\sin\theta_R) = \frac{1}{v_2}E_{0T}\sin\theta_T$$

$$E_{0I} + E_{0R} = E_{0T}$$

$$\frac{1}{\mu_1 v_1}(-E_{0I}\cos\theta_I + E_{0R}\cos\theta_R) = \frac{1}{\mu_2 v_2}(-E_{0T}\cos\theta_T)$$

By a little bit of algebra, it can be shown that both equations 2 and 3 are the same, which means, we have $E_{0I} + E_{0R} = E_{0T}$.

From equation 4 we obtain $\frac{1}{\mu_1 v_1}(E_{0I} - E_{0R})\cos\theta_I = \frac{1}{\mu_2 v_2}E_{0T}\cos\theta_T$. Therefore, we finally obtain $E_{0I} - E_{0R} = \frac{\mu_1 v_1 \cos\theta_T}{\mu_2 v_2 \cos\theta_I}E_{0T} = \alpha\beta E_{0T}$, where $\alpha = \frac{\cos\theta_T}{\cos\theta_I}$ and $\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}$, as defined before.

Fresnel's Equations for Perpendicular Polarization. Solving the previous two equations we finally get

Dr. Avradip Pradhan,
Assistant Professor,
Department of Physics,
Narajole Raj College, Narajole.

$$E_{0R} = \frac{1-\alpha\beta}{1+\alpha\beta} E_{0I}$$

$$E_{0T} = \frac{2}{1+\alpha\beta} E_{0I}$$

These two equations are known as *Fresnel's Equations for Perpendicular Polarization*.

Reflection and Transmission Coefficients. Here, the incident intensity will be $I_I = \frac{1}{2}\varepsilon_1 v_1 {E_{0I}}^2 \cos\theta_I$, the reflected intensity will be $I_R = \frac{1}{2}\varepsilon_1 v_1 {E_{0R}}^2 \cos\theta_R$ and transmitted energy will be $I_T = \frac{1}{2}\varepsilon_2 v_2 {E_{0T}}^2 \cos\theta_T$.

So, the reflection coefficient $R = \frac{I_R}{I_I} = \frac{{E_{0R}}^2}{{E_{0I}}^2} = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2$ and the transmission coefficient $T = \frac{I_T}{I_I} = \frac{\varepsilon_2 v_2 {E_{0T}}^2 \cos\theta_T}{\varepsilon_1 v_1 {E_{0I}}^2 \cos\theta_I} = \alpha\beta\left(\frac{2}{1+\alpha\beta}\right)^2 = \frac{4\alpha\beta}{(1+\alpha\beta)^2}$.

We again see that $R + T = 1$, which establishes the *conservation of energy*.

Brewster's Angle, Brewster's Law:

We see from the Fresnel's Equations for Parallel Polarization that the transmitted wave is always *in phase* with the incident one, but the reflected wave is either *in phase*, if $\alpha > \beta$, or 180^0 *out of phase*, if $\alpha < \beta$ (as shown in Fig. 3, with $n_1 = 1$ and $n_2 = 1.5$). The amplitudes of the transmitted and reflected waves depend on the angle of incidence (θ_I), because α is a function of θ_I.

$$\alpha = \frac{\cos\theta_T}{\cos\theta_I} = \frac{\sqrt{1-\sin^2\theta_T}}{\cos\theta_I} = \frac{\sqrt{1-\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_I}}{\cos\theta_I}$$

In the case of normal incidence ($\theta_I = 0$), we get $\alpha = 1$. At grazing incidence ($\theta_I = \frac{\pi}{2}$), α diverges, and the wave is totally reflected (as seen in Fig. 3). We also see that there is an intermediate angle, θ_B (called *Brewster's Angle*), at which the reflected wave is completely extinguished ($\frac{E_{0R}}{E_{0I}} = 0$).

Dr. Avradip Pradhan,
Assistant Professor,
Department of Physics,
Narajole Raj College, Narajole.

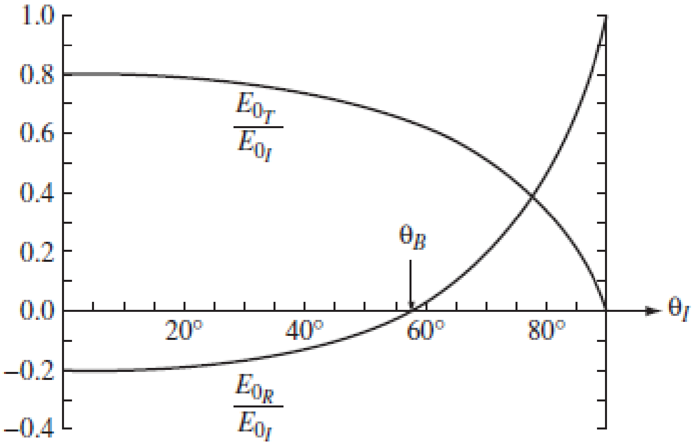

Fig. 3

This happens when $\alpha = \beta$ or $\alpha^2 = \beta^2$ or $1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_B = \beta^2\cos^2\theta_B = \beta^2(1 - \sin^2\theta_B)$ or $\sin^2\theta_B = \frac{1-\beta^2}{\left(\frac{n_1}{n_2}\right)^2 - \beta^2}$.

Usually for most of the media $\mu_1 \approx \mu_2$. So, $\beta \approx \frac{n_2}{n_1}$ or $\sin^2\theta_B \approx \frac{{n_2}^2}{{n_1}^2+{n_2}^2}$.

Therefore, we finally obtain $\tan\theta_B \approx \frac{n_2}{n_1}$. This is the useful formula to calculate Brewster's Angle for two media. For example, with $n_1 = 1$ and $n_2 = 1.5$ we obtain, $\theta_B = \tan^{-1}\frac{n_2}{n_1} \approx 56^0$ (marked in Fig. 3).

Now we see from the Fresnel's Equations for Perpendicular Polarization that both the transmitted wave and reflected wave are always *in phase* with the incident one. So, there is *no crossover* from negative to positive in the values of $\frac{E_{0R}}{E_{0I}}$, and no concept of Brewster's Angle there.

Because waves polarized perpendicular to the plane of incidence exhibit no corresponding quenching of the reflected component, an arbitrary unpolarized beam incident at Brewster's Angle yields a reflected beam that is fully polarized perpendicular to the plane of incidence (or parallel to the interface). This statement is known as *Brewster's Law*.

Dr. Avradip Pradhan,
Assistant Professor,
Department of Physics,
Narajole Raj College, Narajole.

Total Internal Reflection, Evanescent Wave:

According to Snell's law, $\frac{\sin\theta_I}{\sin\theta_T} = \frac{n_2}{n_1}$.

Therefore when light passes from an optically denser medium into a rarer one ($n_1 > n_2$) the propagation vector $\vec{k}$ bends away from the normal (since $\sin\theta_T > \sin\theta_I$ means $\theta_T > \theta_I$).

Particularly, if the light is incident at the *critical angle* (θ_c) defined as $\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$, we obtain $\sin\theta_T = 1$ or $\theta_T = \frac{\pi}{2}$ =90^0, which means, the transmitted ray just grazes the surface. If $\theta_I > \theta_c$ then $\sin\theta_T > 1$, which doesn't correspond to any physically possible θ_T. So, there is no refracted ray at all, rather the whole wave gets reflected back to the denser medium. That's why this is the phenomenon is popularly called as *total internal reflection*.

Although there is no refraction in the rarer medium in total internal reflection, but the fields are not zero in that medium. What we get is a so-called *evanescent wave*, which is a rapidly attenuated wave and it transports no energy into the rarer medium.

A quick way to construct the evanescent wave is done below (shown in Fig. 4).

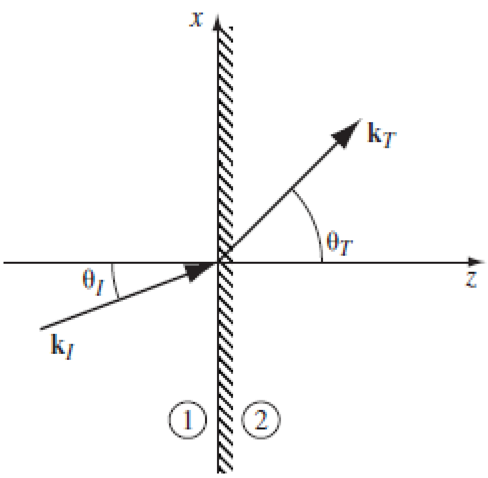

Fig. 4

Dr. Avradip Pradhan,
Assistant Professor,
Department of Physics,
Narajole Raj College, Narajole.

Here we can write the transmitted wave vector as $\vec{k}_T = k_T(\sin\theta_T\,\hat{x} + \cos\theta_T\hat{z})$, with $k_T = \frac{\omega}{v_2} = \frac{\omega n_2}{c}$.

The only change is that, here $\sin\theta_T > 1$ and obviously $\cos\theta_T = \sqrt{1-\sin^2\theta_T}$ will be imaginary.

Now, the transmitted wave can be written as $\vec{E}_T(\vec{r},t) = \vec{E}_{0T}e^{i(\vec{k}_T.\vec{r}-\omega t)}$, with $\vec{k}_T.\vec{r} = k_T\sin\theta_T x + k_T\cos\theta_T z = \frac{\omega n_2}{c}\sin\theta_T x + i\frac{\omega n_2}{c}\sqrt{\sin^2\theta_T - 1}z = \frac{\omega n_1}{c}\sin\theta_I x + i\frac{\omega}{c}\sqrt{{n_1}^2\sin^2\theta_I - {n_2}^2}z = k^* x + i\kappa z$.

Here $k^* = \frac{\omega n_1}{c}\sin\theta_I$ and $\kappa = \frac{\omega}{c}\sqrt{{n_1}^2\sin^2\theta_I - {n_2}^2}$.

Therefore the transmitted wave can be expressed as $\vec{E}_T(\vec{r},t) = \vec{E}_{0T}e^{i(\vec{k}_T.\vec{r}-\omega t)} = \vec{E}_{0T}e^{-\kappa z}e^{i(k^* x-\omega t)}$.

This is the wave, called evanescent wave propagating in the x direction (parallel to the interface), and attenuated in the z direction, with a length scale $\sim\kappa^{-1}$.

Reflection at a Conducting (or Metallic) Surface:

The boundary conditions we used to analyze reflection and refraction at an interface between two dielectrics do not hold in the presence of free charges and free currents. Instead, we have the more general relations

$$\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = \sigma_f \qquad \vec{E}_1^{\parallel} - \vec{E}_2^{\parallel} = 0$$

$$B_1^{\perp} - B_2^{\perp} = 0 \qquad \frac{1}{\mu_1}\vec{B}_1^{\parallel} - \frac{1}{\mu_2}\vec{B}_2^{\parallel} = \vec{K}_f \times \hat{n}$$

where σ_f is the free surface charge, $\vec{K}_f$ is the free surface current, and $\hat{n}$ is a unit vector perpendicular to the surface, pointing from medium 2 into medium 1.

Dr. Avradip Pradhan,
Assistant Professor,
Department of Physics,
Narajole Raj College, Narajole.

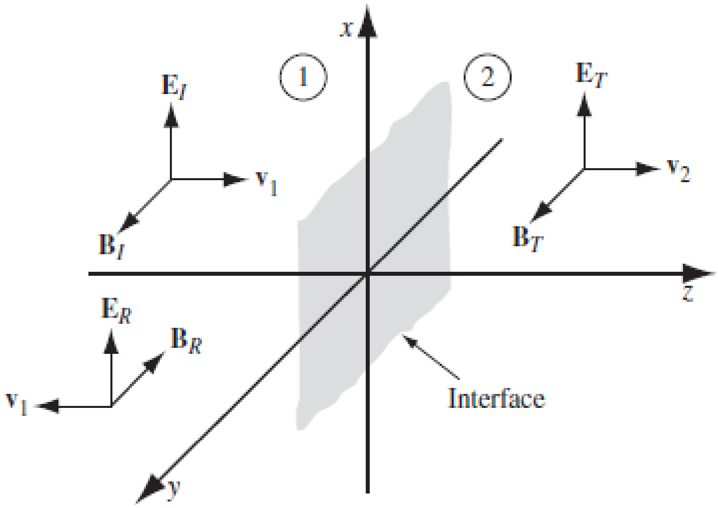

Fig. 5

Suppose now that the $x - y$ plane forms the boundary between a nonconducting linear medium 1 and a conductor or metallic medium (medium 2). A monochromatic plane wave, travelling in the z direction and polarized in the x direction, approaches from the left, as in Fig. 5 can be written as

$$\vec{E}_I(z,t) = E_{0I} e^{i(k_1 z - \omega t)} \hat{x} \text{ and } \vec{B}_I(z,t) = \frac{1}{v_1} E_{0I} e^{i(k_1 z - \omega t)} \hat{y}$$

It creates a reflected wave

$$\vec{E}_R(z,t) = E_{0R} e^{i(-k_1 z - \omega t)} \hat{x} \text{ and } \vec{B}_R(z,t) = -\frac{1}{v_1} E_{0R} e^{i(-k_1 z - \omega t)} \hat{y}$$

and a transmitted wave

$$\vec{E}_T(z,t) = E_{0T} e^{i(\bar{k}_2 z - \omega t)} \hat{x} \text{ and } \vec{B}_T(z,t) = \frac{\bar{k}_2}{\omega} E_{0T} e^{i(\bar{k}_2 z - \omega t)} \hat{y}$$

For transmission into a conducting (or metallic) medium, $\bar{k}_2$ can be shown to be complex. It means, the transmitted wave gets attenuated as it penetrates into the conductor.

At the interface ($z = 0$), the boundary conditions yield (using similar recipe)

$$E_{0R} = \frac{1-\bar{\beta}}{1+\bar{\beta}} E_{0I}$$

Dr. Avradip Pradhan,
Assistant Professor,
Department of Physics,
Narajole Raj College, Narajole.

and $E_{0T} = \frac{2}{1+\bar{\beta}} E_{0I}$

This is the same result derived before with $\alpha = \frac{\cos\theta_T}{\cos\theta_I} = 1$ for normal incidence. There is one more difference. $\bar{\beta} = \frac{\mu_1 v_1}{\mu_2 \omega} \bar{k}_2$ is a complex number here.

For a perfect conductor, $\sigma \to \infty$ and $\bar{k}_2 \to \infty$. So, $\bar{\beta}$ is infinitely large for a perfect conductor or metal. We obtain $E_{0R} = -E_{0I}$ and $E_{0T} = 0$.

In this case the wave is totally reflected, with a 180^0 or π phase shift. That's why excellent conductors make *good mirrors*. For example, if we paint a thin coating of silver onto the back of a pane of glass, the glass starts reflecting. Actually the glass has nothing to do with the reflection. It's just there to support the silver and to keep it from tarnishing.

Reference:

Introduction to Electrodynamics, D.J. Griffiths, Pearson

(All the figures have been collected from the above mentioned reference)