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Topic:

Complex Analysis: Cauchy-Riemann Equation, Analytic Function,Cauchy-Riemann Equation in Polar Form, Poles & Zeroes, Taylor Series, Singular Point and its types

COMPLEX ANALYSIS

Contour:

A contour is a Jordan curve consisting of continuous chain of a finite number of regular arcs. This contour is said to be closed if the starting point A of the arc coincides with the end of the point B of the last arc.

Cauchy-Riemann Equation

Let us consider two functions $u(x, y) and\ v(x, y)$ of real variables(x, y) such that,

$f(z) = u(x, y) + iv(x, y)$ is a function of complex variable$z = x + iy$.

Here,

(i) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ (ii) $-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$

These equations are known as Cauchy-Riemann Equation.

Analytic Function:

A single valued function $f(z)$, differentiable at$z = z_0$, is said to be analytic at $z = z_0$ in the domain D. The point at which the function $f(z)$ is not differentiable is called a singular point of the function.

The necessary conditions for function $f(z) = u + iv$ to be analytic at all points in the domain D, are-

$(i)\ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ (ii) $-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$

Cauchy-Riemann Equation in Polar Form

$(i)\ \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ (ii) $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

Zeros of Analytic function:

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The value of $z$for which the analytic function $f(z)$ becomes zero is said to be zero of $f(z)$.

As for example: Zeroes of z^2-3z+2 are $z = 1 \ and \ 2$

Zeroes of $Cos\, z$ are $\pm(2n-1)\frac{\pi}{2}$, where $n = 1,2,3, \dots \dots \dots \dots$

Cauchy's Integral Formula

If $f(z)$ is analytic within and on the closed curve C and a is a point within C, then

$$f(a)=\frac{1}{2\pi i}\int \frac{f(z)}{z-a}\, dz$$

$$f'(a)=\frac{1}{2\pi i}\int \frac{f(z)}{(z-a)^2}\, dz$$

$$f''(a)=\frac{2!}{2\pi i}\int \frac{f(z)}{(z-a)^3}\, dz$$

$$f^n(a)=\frac{n!}{2\pi i}\int \frac{f(z)}{(z-a)^{n+1}}\, dz$$

Cauchy's Residue Theorem

If a function$f(z)$ is analytic within and on the closed curve C, except at a finite number of poles within C, then

The integral $\int_C \ f(z)dz = 2\pi i \sum R$

Where, $\sum R$ is the sum of the residues at poles within C, the integration being taken in positive.

Taylor Series

$$f(z) = f(a) + (z-a)f'(a) + (z-a)^2\frac{f''(a)}{2!} + \dots\dots + (z-a)^n\frac{f^n(a)}{n!} + \dots\dots$$

Problem-1:

Show that $e^x(Cosy + i\, Siny)$ is a analytic function.

Let, $e^x(Cosy + i\, Siny) = u + iv$

Therefore, $u = \ e^x(Cosy)$ and $v = e^x(\ Siny)$

$\frac{\partial u}{\partial x} = e^x(Cosy); \frac{\partial v}{\partial x} = e^x(\ Siny)$

(ii) $\frac{\partial u}{\partial y} = -e^x(\,Siny);\, \frac{\partial v}{\partial y} = e^x(Cosy)$

Therefore,

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ (ii) $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Thus, Cauchy-Riemann equation is satisfied and hence, $e^x(Cosy + i\, Siny)$ is a analytic function.

Problem-2:

Evaluate the integral:

$$\int_C \frac{e^z(z^2+1)}{(z-1)^2}dz, \text{where C is the circle } |z| = 2$$

Answer: As C is given by $|z| = 2$, the point $z = 1$ lies within C, then according to the theorem

$f'(a) = \frac{1}{2\pi i}\int \frac{f(z)}{(z-a)^2}\, dz$

$\int_C \frac{e^z(z^2+1)}{(z-1)^2}dz = \int_C \frac{f(z)}{(z-1)^2}dz = 2\pi i f'(1)$

Now, $f(z) = e^z(z^2+1)$

Therefore, $f'(z) = e^z(z^2+2z+1) = 4e$

Therefore, $\int_C \frac{f(z)}{(z-1)^2}dz = 2\pi i 4e = 8\pi e i$

Problem-3:

Evaluate the integral:

$$\int_C \frac{e^{-z}}{z+1}dz, \text{where C is the circle } |z| = \frac{1}{2}$$

Poles of the integrand are given by putting the denominator equal to zero.

So, $z + 1 = 0$

Therefore, $z = -1$

The given circle $|z| = \frac{1}{2}$ with the centre at $z = 0$ and radius $\frac{1}{2}$

The point $z = -1$ lies outside the circle $|z| = \frac{1}{2}$

Therefore, the function $\frac{e^{-z}}{z+1}$ is analytic within and on C

By Cauchy's integral theorem $\int_C \frac{e^{-z}}{z+1} dz = 0$

Problem-4:

Use Cauchy's Integral formula to evaluate:

$$\int_C \frac{2z+1}{z^2+z} dz, \text{ where C is the circle } |z| = \frac{1}{2}$$

Hence, $z^2 + z = 0$

Therefore, $z(z+1) = 0$

So, $z = 0, -1$

So, $|z| = \frac{1}{2}$ is a circle with centre at origin and radius=$\frac{1}{2}$

Therefore, it encloses only one pole at $z = 0$

Therefore, $\int_C \frac{2z+1}{z^2+z} dz = \int_C \frac{\frac{2z+1}{z+1}}{z} dz = 2\pi i \left[\frac{2z+1}{z+1}\right]_{z=0} = 2\pi i$

Singular Point:

A point at which a function $f(z)$ is not analytic is known as a singular point or singularity of the function.

For example, the function $\frac{1}{z-2}$ has singular point at $z - 2 = 0$ or $z = 2$

Isolated Singular Point: If $z = a$ is a singularity of $f(z)$ and if there is no other singularity within a small circle surrounding the point $z = a$, then $z = a$ is said to be an isolated singularity of the function $f(z)$; otherwise it is called non-isolated.

The function $\frac{1}{(z-1)(z-3)}$ has two isolated singular points, namely $z = 1$ and $z = 3$

Because, $(z-1)(z-3) = 0$

Therefore, $z = 1$ and $z = 3$

Non-isolated Singular Point: If $z = a$ is a singularity of $f(z)$ and if there is no other singularity within a small circle surrounding the point $z = a$, then $z = a$ is said to be an isolated singularity of the function $f(z)$; otherwise it is called non-isolated.

For example, $\frac{1}{\text{Sin}\frac{\pi}{z}}$ is not analytic at the points where, $\text{Sin}\ \frac{\pi}{z} = 0$ at the points $\frac{\pi}{z} = n\pi$

Therefore, z =$\frac{1}{n}$, where, n=1,2,3……..

Thus, z =1,$\frac{1}{2}$,$\frac{1}{3}$, … … … … ….

Pole of order m: Let a function $f(z)$ have an isolated singular point $z = a$, $f(z)$ can be expanded in a Laurent's series around $z = a$, giving

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots\ldots + \frac{b_1}{(z-a)} + \frac{b_1}{(z-a)^2} + \cdots \ldots \ldots + \frac{b_m}{(z-a)^m}$$

Here, $z = a$ is said to be a pole of order m of function $f(z)$, when m=1, the pole is said to be simple pole. In this case,

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots\ldots + \frac{b_1}{(z-a)}$$

If the number of the terms of negative powers in expansion is infinite, then $z = a$ is called an essential singular point of $f(z)$.

Problem-5:

Find the pole of $f(z) = Sin(\frac{1}{z-a})$

$$Sin\left(\frac{1}{z-a}\right) = \frac{1}{z-a} - \frac{1}{3!(z-a)^3} + \frac{1}{5!(z-a)^5} - \ldots\ldots\ldots$$

The given function $f(z)$ has infinite number of terms in the negative powers of $z - a$

So, $f(z)$ has essential singularity at $z = a$.

Problem-5:

Find the pole of $f(z) = \frac{Sin\,(z-a)}{(z-a)^4}$

$$\frac{Sin\,(z-a)}{(z-a)^4} = \frac{1}{(z-a)^4}\left[(z-a) - \frac{(z-a)^3}{3!} + \frac{(z-a)^5}{5!} - \frac{(z-a)^7}{7!} + \ldots\ldots\right]$$

$$=\frac{1}{(z-a)^3}[1-\frac{(z-a)^2}{3!}+\frac{(z-a)^4}{5!}-\frac{(z-a)^6}{7!}+.......]$$

Therefore, given function has a negative power 3 of $(z-a)$

So, $f(z)$ has a pole at $z=a$ of order 3.

Problem-6:

Discuss the singularityof $\frac{1}{1-e^z}$ at $\mathbf{z}=2\pi i$

We have $f(z)=\frac{1}{1-e^z}$

The poles are determined by putting the denominator equal to zero.

i.e. $1-e^z=0$

Therefore, $e^z=1=(Cos\ 2n\pi+i\ Sin\ 2n\pi)=e^{2n\pi i}$

Therefore, $z=2n\pi i$ (where, $n=0,\pm1,\ \pm2,\dots\dots.)$

Clearly, $z=2\pi i$ is a simple pole.

Frequently Asked Questions/Numerical:

For more problems in this section of Complex Analysis, students can solve the problems of Mathematical Physics: Author-H.K. Dass & Rama Verma, Published by S. Chand (2018 Ed.) and Principles of Mathematical Physics, Author- S. P. Kuila, Published by NCBA (2018 Ed.).

References:

(i) *Principles of Mathematical Physics, Author- S. P. Kuila, Published by NCBA (2018 Ed.).*
(ii) *Mathematical Physics: Author-H.K. Dass & Rama Verma,Published by S. Chand(2018 Ed.).*

Link to Audio visual Lectures (e-Lectures) given by Distinguish Professors of Foreign Universities:

(i) Part I: Complex Variables, Lecture 1: The Complex Numbers Instructor: Herbert Gross View the complete course: http://ocw.mit.edu/RES18-008F11
(ii) Cauchy Integral Formula: https://www.youtube.com/watch?v=WJOf4PfoHow
(iii) https://www.youtube.com/watch?v=K5DeHvUKzK4

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