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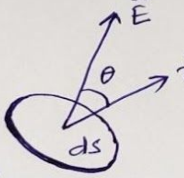
Paper: C3T (Electricity and Magnetism)

Topic: Electric Flux and Gauss's Law

Flux of Electric Field : Let us consider a surface element $d\vec{s} = ds\,\hat{n}$ in an electric field $\vec{E}$. Here, $\hat{n}$ is the outward unit vector normal to the surface element.



The quantity
$$d\Phi = \vec{E}.d\vec{s}$$
$$= E\cos\theta\, ds$$
is called the flux of $\vec{E}$ through $d\vec{s}$.

The flux of $\vec{E}$ over any arbitrary surface S is given by
$$\Phi = \iint_S \vec{E}.d\vec{s}$$

Gauss's law : It's a law that describes the relationship between the total electric flux over a closed surface and the total electric charge enclosed by the surface. Considering SI units, the law may be stated as follows :

In an arbitrary electrostatic field in vacuum, the total ~~ee~~ electric flux over any closed surface is equal to $\frac{1}{\epsilon_0}$ times the total charge ~~et~~ enclosed by the surface, where ϵ_0 is the permittivity of free space.

For a closed surface S enclosing N number of point charges, say : $q_1, q_2, \ldots q_N$, the ~~law~~ Gauss's law says
$$\oint_S \vec{E}.d\vec{s} = \frac{1}{\epsilon_0}\sum_{i=1}^{N} q_i$$

For a continuous distribution of charge within a volume V with charge density ρ :
$$\oint_S \vec{E}.d\vec{s} = \frac{1}{\epsilon_0}\iiint_V \rho\, dV$$

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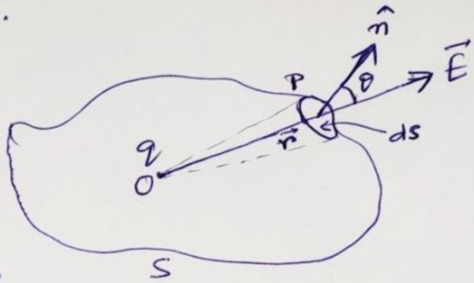
Proof of Gauss's law: Let's consider a point charge q located at the point 'O' and let's draw an arbitrary close surface S enclosing the point charge as shown below.



Let's consider a surface element $d\vec{s} = \hat{n}\, ds$ around a point P on the surface S. Electric field at point P is

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \cdot \frac{\hat{r}}{r^2} \qquad \text{—— (1)}$$

The flux of $\vec{E}$ through the close surface S is

$$\oiint_S \vec{E} \cdot d\vec{s} = \frac{q}{4\pi\epsilon_0} \oiint_S \frac{\hat{r}\cdot\hat{n}}{r^2}\, ds \qquad \text{—— (2)}$$

Now, the quantity $\frac{\hat{r}\cdot\hat{n}\, ds}{r^2} = \frac{ds \cos\theta}{r^2} = d\Omega$ is the solid angle subtended by ds at q. Therefore,

$$\oiint_S \vec{E}\cdot\hat{n}\, ds = \frac{q}{4\pi\epsilon_0} \iint d\Omega$$

$$= \frac{q}{4\pi\epsilon_0} \cdot 4\pi$$

$$= \frac{q}{\epsilon_0} \qquad \text{—— (3)}$$

There may be a case where some charges may lie outside the surface S. We will now show that flux over a closed surface due to charges outside will be zero.

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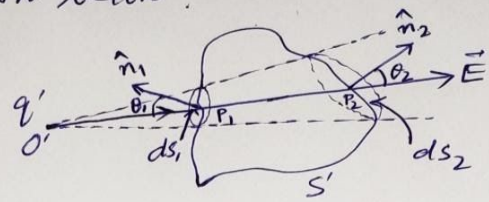
Consider a charge q' at a point O' outside a closed surface S' as shown below:



The electric field lines enter the close surface through $d\vec{s}_1$ and leave it through $d\vec{s}_2$. The solid angles subtended by $d\vec{s}_1$ and $d\vec{s}_2$ at O' are the same. So, the electric flux through $d\vec{s}_1$ and $d\vec{s}_2$ will be equal and opposite and hence they contribute nothing to the total electric flux over S'. This is true for all cones drawn from O' to cover the whole surface S'. Therefore, the contribution to the total electric flux over a close surface S will be zero when charges lie outside the surface.

If there are a number of charges $q_1, q_2, \ldots, q_N$ inside the surface, then by the principle of superposition the total field $\vec{E} = \sum_{i=1}^{N} E_i$; where $\vec{E}_i$ is the field due to i-th charge q_i. The total flux for all the point charges would be

$$\oint\!\!\oint_S \vec{E}\cdot\hat{n}\,ds = \oint\!\!\oint_S (\vec{E}_1 + \vec{E}_2 + \cdots + \vec{E}_N)\cdot\hat{n}\,ds$$

$$= \frac{q_1}{4\pi\epsilon_0}\oint\!\!\oint_S d\Omega_1 + \frac{q_2}{4\pi\epsilon_0}\oint\!\!\oint_S d\Omega_2 + \cdots$$

$$= \frac{q_1}{\epsilon_0} + \frac{q_2}{\epsilon_0} + \cdots + \frac{q_N}{\epsilon_0}$$

$$= \frac{1}{\epsilon_0}\sum_{i=1}^{N} q_i \qquad \text{--- (4)}$$

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Since the contribution of the charges ~~which~~ lying outside the surface to the net flux is zero, eqn. 4 can be considered as the total flux due to all the charges lying within as well as outside the surface. We can generalize the eqn.-4 for the case of continuous charge distribution. If the surface S encloses a volume V and ρ be the charge density within V then eqn 4 becomes

$$\oiint_S \vec{E} \cdot \hat{n}\, ds = \frac{1}{\epsilon_0} \iiint_V \rho \, dV \qquad \text{— } 5$$

Eqn. 5 is also called integral form of Gauss's law.

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Differential form of Gauss's law:

We have the integral form of Gauss's law as following:

$$\oiint_S \vec{E}\cdot\hat{n}\, ds = \frac{1}{\epsilon_0}\iiint_V \rho\, dV$$

According to the Gauss's divergence theorem

$$\oiint_S \vec{A}\cdot\hat{n}\, ds = \iiint_V \vec{\nabla}\cdot\vec{A}\, dV \, .$$

Applying this divergence theorem to the left hand side of the integral form, we get

$$\iiint_V \vec{\nabla}\cdot\vec{E}\, dV = \frac{1}{\epsilon_0}\iiint_V \rho\, dV \, .$$

Since the above eqn. is true for any volume V, the integrands must be equal. Therefore,

$$\boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}} \qquad \text{--- (6)}$$

This is the differential form of Gauss's law. It's an important equation in electrostatics which relates electric field at a point with the charge density at that point.

Since $\vec{E} = -\vec{\nabla}\phi$, eqn. (6) may be written as

$$\vec{\nabla}\cdot(-\vec{\nabla}\phi) = \frac{\rho}{\epsilon_0}$$

or, $$\boxed{\nabla^2\phi = -\frac{\rho}{\epsilon_0}} \qquad \text{--- (7)}$$

This equation is known as Poisson's equation.

When the charge density $\rho = 0$, eqn. (7) reduces to

$$\boxed{\nabla^2\phi = 0} \qquad \text{--- (8)}$$

This is known as Laplace's equation.