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Topic:

Eigen Values & Eigen Vectors: Cayley- Hamiliton Theorem. Eigen values & Eigen vectors, Diagonalization of Matrices.

EIGEN VALUES & EIGEN VECTORS

Caley-Hamilton Theorem:

Statement: Every square matrix satisfies its own characteristics equation.

If A be the square matrix of order n, the characteristics polynomial becomes-

$|A-\lambda I| = a_0 + a_1\lambda + a_2\lambda^2 + \dots\dots\dots + a_n\lambda^n$

The matrix equation becomes-

$a_0 I + a_1 X + a_2 X^2 + \dots\dots\dots + a_n X^n = 0$ is satisfied by $X=A$

If A be a square matrix with characteristics equation,

$\lambda^3 - 2\lambda^2 + 3\lambda - 4 = 0$

then according to Caley-Hamilton Theorem, we may write,-

$A^3 - 2A^2 + 3A - 4 = 0$

From this equation we may found A^{-1}

Multiplying both sides by A^{-1}, we get

$A^{-1}A^3 - 2A^{-1}A^2 + 3A^{-1}A - 4A^{-1} = 0$

or, $A^2 - 2A + 3I - 4A^{-1} = 0$

$A^{-1} = \frac{1}{4}[A^2 - 2A + 3I]$

Let us now consider Caley-Hamilton theorem by an example-

(A) Verify Caley-Hamilton Theorem for the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$. **Find** A^{-1} **and determine** A^8.

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The characteristics equation of the given matrix A is

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

Or, $(\lambda-1)(\lambda+1)-4=0$

Or, $\lambda^2-5=0$

Now, $A^2 = A.A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 5I$

Or, $A^2-5I=0$

Multiplying both sides by A^{-1}, we get

$A^{-1}A^2 - 5A^{-1}I=0$

or, $A-5A^{-1} = 0$

$A^{-1} = \frac{1}{5}A = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

To find A^8, we multiply $A^2-5I=0$ by A^6

Therefore, $A^6 A^2-5I A^6=0$

Or, $A^8=5A^6=5A^2.A^2.A^2 = 5(5I)(5I)(5I)=625I$

Eigen values or Characteristics Roots: If, A be the square matrix and λ, a scalar, then the matrix $A-\lambda I$ is called the characteristic matrix of A, where I is the unit matrix.

$|A-\lambda I|=0$ is called the characteristic equation of matrix.

The roots of the characteristic equation, $|A-\lambda I|=0$ is called the Eigen values or characteristics roots of matrix A.

Let us study by an example-

Now, we consider a matrix $[A] = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

(a) The characteristic matrix of A is-

$$A-\lambda I = \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}$$

(b) The characteristic polynomial-

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)[(3-\lambda)(2-\lambda)] - 2[(2-\lambda)-1] + 1[2-(3-\lambda)]$$

Or, $-\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0$

Or, $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

(c) The characteristic equation-

$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

(d) The characteristic values/Eigen values-

$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

or, $(\lambda-5)(\lambda-1)(\lambda-1)=0$

i.e., $\lambda_1=5, \lambda_2=1, \lambda_3=1$

Properties of Eigen Values:

i. Any square matrix A and its Transpose A′ have same Eigen values.
ii. The sum of the Eigen values of the matrix is equal to the trace of that matrix.
iii. The product of the Eigen values of the matrix is equal to the determinantof that matrix.
iv. If $\lambda_1, \lambda_2, \lambda_3$ are Eigen values of matrix A, then, Eigen values of -
a) kA are $k\lambda_1, k\lambda_2, k\lambda_3$
b) A^m are $\lambda_1^m, \lambda_2^m, \lambda_3^m$
c) A^{-1} are $1/\lambda_1, 1/\lambda_2, 1/\lambda_3$

Characteristics Vector or Eigenvectors:

For each value of the eigenvalues λ, there is a non-zero vector X satisfying the equation,

$$|A-\lambda I| X=0$$

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The non-zero value X is called the characteristics vector or Eigenvector.

$X_1=\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, X_2=\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}, X_3=\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$

Diagonalization of a Matrix:

If a square matrix A of order n has n linearly independent eigenvectors, then a matrix P can be found out such that $P^{-1}AP$ is a diagonal matrix.

Where, $[P]= X_1=\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$ $A=\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

$P^{-1}AP=D$

Where, D $==\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

In single sentence, you have to put the Eigen values of the matrix diagonally and made a diagonal matrix by putting remaining elements to 0.

Frequently Asked Questions:

a. *Determine the Eigen valuesand normalized Eigen vectors of the following matrix*

$A=\begin{bmatrix} 3 & 5 & 1 \\ -2 & -2 & 0 \\ 2 & 7 & 3 \end{bmatrix}$.

b. *Determine the Eigen values and normalized Eigen vectors of the following hermitianmatrix*

$A=\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

c. *Verify Caley Hamilton theorem for the matrix* $A=\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$. *Find* A^{-1} *hence find* A^8.

d. *Verify Caley Hamilton theorem for the matrix* $A=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$. *Find* A^{-1} *hence find* A^8.

e. *Diagonalize* $A=\begin{bmatrix} Cos\beta & -Sin\beta \\ Sin\beta & Cos\beta \end{bmatrix}$.

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f. *Determine Eigen Value & Eigen Vectors of the matrix* $A=\begin{bmatrix} 3 & i \\ -i & 3 \end{bmatrix}$. *Hence Diagonalize the matrix.*

g. $\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 6x = 0$, $x(0)=1$, $x'(0)=2$; *Solve this differential equation by matrix method.*

References:

(i) *Principles of Mathematical Physics, Author- S. P. Kuila, Published by NCBA (2018 Ed.).*
(ii) *Mathematical Physics: Author-H.K. Dass& Rama Verma,Published by S. Chand(2018 Ed.).*