LECTURE NOTES ON TRIPLE PRODUCT

PROF. SHILPA PATRA

Department of Mathematics
Narajole Raj College

1. Triple Product

In vector algebra, a branch of mathematics, the triple product is a product of three 3-dimensional vectors, usually Euclidean vectors. The name "triple product" is used for two different products, the scalar-valued scalar triple product and the vector-valued vector triple product.

2. Scalar triple product

The scalar triple product (also called the mixed product, box product, or triple scalar product) is defined as the dot product of one of the vectors with the cross product of the other two.

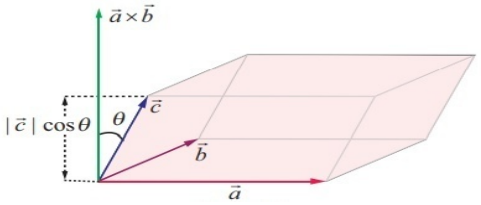
Definition 2.1. *For a given set of three vectors* $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$, *the scalar* $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ *is called a scalar triple product of* $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$.

Remark: $\mathbf{a}\cdot\mathbf{b}$ is a scalar, and so $(\mathbf{a}\cdot\mathbf{b})\times\mathbf{c}$ has no meaning.

Note: Given any three vectors $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$, the following are scalar triple products:

$$\begin{aligned}&(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c},\ (\mathbf{b}\times\mathbf{c})\cdot\mathbf{a},\ (\mathbf{c}\times\mathbf{a})\cdot\mathbf{b},\ \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}),\ \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}),\ \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})\\&(\mathbf{b}\times\mathbf{a})\cdot\mathbf{c},\ (\mathbf{c}\times\mathbf{b})\cdot\mathbf{a},\ (\mathbf{a}\times\mathbf{c})\cdot\mathbf{b},\ \mathbf{a}\cdot(\mathbf{c}\times\mathbf{b}),\ \mathbf{b}\cdot(\mathbf{a}\times\mathbf{c}),\ \mathbf{c}\cdot(\mathbf{b}\times\mathbf{a})\end{aligned}\tag{2.1}$$

Geometric interpretation of scalar triple product:



Geometrically, the scalar triple product $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ is the (signed) volume of the parallelepiped defined by the three vectors given. Indeed, the magnitude of the vector $\mathbf{a} \times \mathbf{b}$ is the area of the parallelogram formed by using $\mathbf{a}$ and $\mathbf{b}$; and the direction of the vector $\mathbf{a} \times \mathbf{b}$ is perpendicular to the plane parallel to both $\mathbf{a}$ and $\mathbf{b}$.

Therefore, $|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|$ is $|\mathbf{a} \times \mathbf{b}||\mathbf{c}||\cos(\theta)|$, where θ is the angle between $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c}$. From the above figure, we observe that $|\mathbf{c}||\cos(\theta)|$ is the height of the parallelepiped formed by using the three vectors as adjacent vectors. Thus, $|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|$ is the volume of the parallelepiped.

Theorem 2.2. *If* $\mathbf{a} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$, *and* $\mathbf{c} = c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}$, *then*

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Proof. By definition, we have

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}\cdot\mathbf{c}\\ &= [(a_2b_3-a_3b_2)\mathbf{i}+(a_3b_1-a_1b_3)\mathbf{j}+(a_1b_2-a_2b_1)\mathbf{k}]\cdot(c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k})\\ &= (a_2b_3-a_3b_2)c_1+(a_3b_1-a_1b_3)c_2+(a_1b_2-a_2b_1)c_3\\ &= \begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}.\end{aligned}$$

□

Theorem 2.3. *For a given set of three vectors* **a**, **b**, *and* **c**,

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}).$$

Proof.

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a} &= \begin{vmatrix}b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\\ a_1 & a_2 & a_3\end{vmatrix}\\ &= -\begin{vmatrix}a_1 & a_2 & a_3\\ c_1 & c_2 & c_3\\ b_1 & b_2 & b_3\end{vmatrix} \text{ by } R_1\leftrightarrow R_3\\ &= \begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix} \text{ by } R_2\leftrightarrow R_3\\ &= (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}\end{aligned}$$

□

Note: By Theorem 2.3, it follows that, in a scalar triple product, dot and cross can be interchanged without altering the order of occurrences of the vectors. For instance, we have

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} &= \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}), \textit{Since dot and cross can be interchanged.}\\ &= (\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}, \textit{Since dot product is commutative.}\\ &= \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}), \textit{Since dot and cross can be interchanged.}\\ &= (\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}, \textit{Since dot product is commutative.}\\ &= \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}), \textit{Since dot and cross can be interchanged.}\end{aligned}$$

Notation: For any three vectors **a**, **b**, *and* **c**, the scalar triple product $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ is denoted by $[\mathbf{a},\mathbf{b},\mathbf{c}]$ is read as box a, b, c . For this reason and also because the absolute value of a scalar triple product represents the volume of a box (rectangular parallelepiped),a scalar triple product is also called a box product.

Note: By the above, we have

$$[\mathbf{a},\mathbf{b},\mathbf{c}]=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=[\mathbf{b},\mathbf{c},\mathbf{a}]$$

$$[\mathbf{b},\mathbf{c},\mathbf{a}]=(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=(\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}=[\mathbf{c},\mathbf{a},\mathbf{b}].$$

In other words, $[\mathbf{a},\mathbf{b},\mathbf{c}] = [\mathbf{b},\mathbf{c},\mathbf{a}] = [\mathbf{c},\mathbf{a},\mathbf{b}]$; that is, if the three vectors are permuted in the same cyclic order, the value of the scalar triple product remains the same.

If any two vectors are interchanged in their position in a scalar triple product, then the value of the scalar triple product is (-1) times the original value. More explicitly,

$$[\mathbf{a},\mathbf{b},\mathbf{c}] = [\mathbf{b},\mathbf{c},\mathbf{a}] = [\mathbf{c},\mathbf{a},\mathbf{b}] = -[\mathbf{a},\mathbf{c},\mathbf{b}] = -[\mathbf{c},\mathbf{b},\mathbf{a}] = -[\mathbf{b},\mathbf{a},\mathbf{c}].$$

Theorem 2.4. *The scalar triple product preserves addition and scalar multiplication. That is,*

$$\begin{aligned}
[\mathbf{a}+\mathbf{b},\mathbf{c},\mathbf{d}] &= [\mathbf{a},\mathbf{c},\mathbf{d}] + [\mathbf{b},\mathbf{c},\mathbf{d}];\\
[\alpha\mathbf{a},\mathbf{b},\mathbf{c}] &= \alpha[\mathbf{a},\mathbf{b},\mathbf{c}];\ \forall\alpha\in\mathbb{R}\\
[\mathbf{a},\mathbf{b}+\mathbf{c},\mathbf{d}] &= [\mathbf{a},\mathbf{b},\mathbf{d}] + [\mathbf{a},\mathbf{c},\mathbf{d}];\\
[\mathbf{a},\alpha\mathbf{b},\mathbf{c}] &= \alpha[\mathbf{a},\mathbf{b},\mathbf{c}];\ \forall\alpha\in\mathbb{R}\\
[\mathbf{a},\mathbf{b},\mathbf{c}+\mathbf{d}] &= [\mathbf{a},\mathbf{b},\mathbf{c}] + [\mathbf{a},\mathbf{b},\mathbf{d}];\\
[\mathbf{a},\mathbf{b},\alpha\mathbf{c}] &= \alpha[\mathbf{a},\mathbf{b},\mathbf{c}];\ \forall\alpha\in\mathbb{R}
\end{aligned}$$

Theorem 2.5. *The scalar triple product of three non-zero vectors is zero if, and only if, the three vectors are coplanar.*

Proof. Let$\mathbf{a},\mathbf{b},\mathbf{c}$ be any three non-zero vectors. Then,

$$\begin{aligned}
(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = 0 \quad &\iff \quad \mathbf{a}\times\mathbf{b} = \mathbf{0}\ or\ (\mathbf{c}\ and\ \mathbf{a}\times\mathbf{b}\ are\ perpendicular)\\
&\iff \quad (\mathbf{a}\ and\ \mathbf{b}\ are\ parallel)\ or\ (\mathbf{a},\ \mathbf{b},\ and\ \mathbf{c}\ are\ coplanar)\\
&\iff \quad \mathbf{a},\ \mathbf{b},\ and\ \mathbf{c}\ are\ coplanar.
\end{aligned}$$

□

Example 2.6. *If* $2\mathbf{i}-\mathbf{j}+3\mathbf{k}$, $3\mathbf{i}+2\mathbf{j}+\mathbf{k}$, $\mathbf{i}+m\mathbf{j}+4\mathbf{k}$ *are coplanar, find the value of* m .

Proof. Since the given three vectors are coplanar, we have

$$\begin{vmatrix} 2 & -1 & 3\\ 3 & 2 & 1\\ 1 & m & 4 \end{vmatrix} = 0 \implies m = -3$$

□

Example 2.7. *Show that the four points (6, 7, 0), (16, 19, 4), (0, 3, 6), (2, 5,10) lie on a same plane.*

Proof. Let A = (6, 7, 0), B = (16, 19, 4), C = (0, 3, 6), D = (2, 5,10) . To show that the four points A, B, C, D lie on a plane, we have to prove that the three vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar.
Now,

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (16\mathbf{i}-19\mathbf{j}-4\mathbf{k}) - (6\mathbf{i}-7\mathbf{j}) = (10\mathbf{i}-12j-4\mathbf{k})$$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (-6\mathbf{i}+10\mathbf{j}-6\mathbf{k})$ and $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = (-4\mathbf{i}+2\mathbf{j}+10\mathbf{k})$.
we have

$$[\overrightarrow{AB},\overrightarrow{AC},\overrightarrow{AD}] = \begin{vmatrix} 10 & -12 & -4\\ -6 & 10 & -6\\ -4 & 2 & 10 \end{vmatrix} = 0.$$

Therefore, the three vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar and hence the four points A, B, C, and D lie on a plane. □

Example 2.8. *If the vectors* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *are coplanar, then prove that the vectors* $\mathbf{a}+\mathbf{b}, \mathbf{b}+\mathbf{c}, \mathbf{c}+\mathbf{a}$ *are also coplanar.*

Proof. Since the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are coplanar, we have $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = 0$ Using the properties of the scalar triple product, we get

$$\begin{aligned}
[\mathbf{a}+\mathbf{b}, \mathbf{b}+\mathbf{c}, \mathbf{c}+\mathbf{a}] &= [\mathbf{a}, \mathbf{b}+\mathbf{c}, \mathbf{c}+\mathbf{a}] + [\mathbf{b}, \mathbf{b}+\mathbf{c}, \mathbf{c}+\mathbf{a}] \\
&= [\mathbf{a}, \mathbf{b}, \mathbf{c}+\mathbf{a}] + [\mathbf{a}, \mathbf{c}, \mathbf{c}+\mathbf{a}] + [\mathbf{b}, \mathbf{b}, \mathbf{c}+\mathbf{a}] + [\mathbf{b}, \mathbf{c}, \mathbf{c}+\mathbf{a}] \\
&= [\mathbf{a}, \mathbf{b}, \mathbf{c}] + [\mathbf{a}, \mathbf{b}, \mathbf{a}] + [\mathbf{a}, \mathbf{c}, \mathbf{c}] + [\mathbf{a}, \mathbf{c}, \mathbf{a}] + [\mathbf{b}, \mathbf{b}, \mathbf{c}] + [\mathbf{b}, \mathbf{b}, \mathbf{a}] + [\mathbf{b}, \mathbf{c}, \mathbf{c}] + [\mathbf{b}, \mathbf{c}, \mathbf{a}] \\
&= [\mathbf{a}, \mathbf{b}, \mathbf{c}] + [\mathbf{a}, \mathbf{b}, \mathbf{c}] = 2[\mathbf{a}, \mathbf{b}, \mathbf{c}] = 0.
\end{aligned}$$

Hence the vectors $\mathbf{a}+\mathbf{b}, \mathbf{b}+\mathbf{c}, \mathbf{c}+\mathbf{a}$ are coplanar. □

Theorem 2.9. *If* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *and* $\mathbf{p}, \mathbf{q}, \mathbf{r}$ *be any systems of three vectors, and if* $\mathbf{p} = x_1\mathbf{a} + y_1\mathbf{b} + z_1\mathbf{c}$, $\mathbf{q} = x_2\mathbf{a} + y_2\mathbf{b} + z_2\mathbf{c}$, *and* $\mathbf{r} = x_3\mathbf{a} + y_3\mathbf{b} + z_3\mathbf{c}$, *then*

$$[\mathbf{p}, \mathbf{q}, \mathbf{r}] = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} [\mathbf{a}, \mathbf{b}, \mathbf{c}].$$

Proof. Applying the distributive law of cross product and using

$$\mathbf{a} \times \mathbf{a} = \mathbf{b} \times \mathbf{b} = \mathbf{c} \times \mathbf{c} = 0$$

$$\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b},\ \mathbf{a} \times \mathbf{c} = -\mathbf{c} \times \mathbf{a},\ \mathbf{c} \times \mathbf{b} = -\mathbf{b} \times \mathbf{c}.$$

We get

$$\begin{aligned}
\mathbf{p} \times \mathbf{q} &= (x_1\mathbf{a} + y_1\mathbf{b} + z_1\mathbf{c}) \times (x_2\mathbf{a} + y_2\mathbf{b} + z_2\mathbf{c}) \\
&= (x_1y_2 - x_2y_1)(\mathbf{a} \times \mathbf{b}) + (y_1z_2 - y_2z_1)(\mathbf{b} \times \mathbf{c}) + (z_1x_2 - z_2x_1)(\mathbf{c} \times \mathbf{a}) \\
&= \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} (\mathbf{a} \times \mathbf{b}) + \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} (\mathbf{b} \times \mathbf{c}) + \begin{vmatrix} z_1 & z_2 \\ x_1 & x_2 \end{vmatrix} (\mathbf{c} \times \mathbf{a}).
\end{aligned}$$

Hence, we get

$$\begin{aligned}
[\mathbf{p}, \mathbf{q}, \mathbf{r}] &= (\mathbf{p} \times \mathbf{q}) \cdot (x_3\mathbf{a} + y_3\mathbf{b} + z_3\mathbf{c}) \\
&= \left\{ x_3 \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} + y_3 \begin{vmatrix} z_1 & z_2 \\ x_1 & x_2 \end{vmatrix} + z_3 \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \right\} [\mathbf{a}, \mathbf{b}, \mathbf{c}] \\
&= \left\{ x_3 \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} + y_3 \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix} + z_3 \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \right\} [\mathbf{a}, \mathbf{b}, \mathbf{c}] \\
&= \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} [\mathbf{a}, \mathbf{b}, \mathbf{c}].
\end{aligned}$$

□

Example 2.10. *If* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *are three vectors, prove that* $[\mathbf{a}+\mathbf{c}, \mathbf{a}+\mathbf{b}, \mathbf{a}+\mathbf{b}+\mathbf{c}] = -[\mathbf{a}, \mathbf{b}, \mathbf{c}]$.

Proof. Using theorem 2.9, we get

$$\begin{aligned}[\mathbf{a}+\mathbf{c}, \mathbf{a}+\mathbf{b}, \mathbf{a}+\mathbf{b}+\mathbf{c}] &= \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} [\mathbf{a}, \mathbf{b}, \mathbf{c}] \\ &= -[\mathbf{a}, \mathbf{b}, \mathbf{c}].\end{aligned}$$

□

Properties(Summary):

- The scalar triple product is unchanged under a circular shift of its three operands (a, b, c):
$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}).$$
- Swapping the positions of the operators without re-ordering the operands leaves the triple product unchanged. This follows from the preceding property and the commutative property of the dot product.
$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$
- Swapping any two of the three operands negates the triple product. This follows from the circular-shift property and the anticommutativity of the cross product.
$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) = -\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) = -\mathbf{c} \cdot (\mathbf{b} \times \mathbf{a}).$$
- The scalar triple product can also be understood as the determinant of the 3×3 matrix that has the three vectors either as its rows or its columns
$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \det(\mathbf{a}, \mathbf{b}, \mathbf{c}).$$
- If the scalar triple product is equal to zero, then the three vectors a, b, and c are coplanar, since the parallelepiped defined by them would be flat and have no volume.
- If any two vectors in the scalar triple product are equal, then its value is zero:
$$\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{b}) = (\mathbf{b} \times \mathbf{a}) \cdot \mathbf{a} = 0.$$

3. Vector triple product

The vector triple product is defined as the cross product of one vector with the cross product of the other two.

Definition 3.1. *The vector triple product of* $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *is* $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$.

Caution: The vector triple product is not associative, i.e. in general

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}.$$

To see this, we note that $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ is perpendicular to $\mathbf{u} \times \mathbf{v}$ which is normal to a plane determined by $\mathbf{u}$ and $\mathbf{v}$. So, $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ is coplanar with $\mathbf{u}$ and $\mathbf{v}$. By the same argument, $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ is coplanar with $\mathbf{v}$ and $\mathbf{w}$. For this reason it is vital that we include the parentheses in a vector triple product to indicate which vector product should be performed first.

We now obtain a formula for the vector triple product which reflects the fact that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$, as it is coplanar with $\mathbf{v}$ and $\mathbf{w}$, may be expressed as $\alpha\mathbf{v} + \beta\mathbf{w}$ for some α, $\beta \in \mathbb{R}$.

Theorem 3.2. *(The triple product expansion, or Lagrange's formula): For all vectors* $\mathbf{u}, \mathbf{v}$, *and* $\mathbf{w}$

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}.$$

Proof. Let $\mathbf{u} = (u_x, u_y, u_z)$, $\mathbf{v} = (v_x, v_y, v_z)$, and $\mathbf{w} = (w_x, w_y, w_z)$.

The x component of $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ is given by:

$$\begin{aligned}
(\mathbf{u} \times (\mathbf{v} \times \mathbf{w}))_x &= \mathbf{u}_y(\mathbf{v}_x\mathbf{w}_y - \mathbf{v}_y\mathbf{w}_x) - \mathbf{u}_z(\mathbf{v}_z\mathbf{w}_x - \mathbf{v}_x\mathbf{w}_z) \\
&= \mathbf{v}_x(\mathbf{u}_y\mathbf{w}_y + \mathbf{u}_z\mathbf{w}_z) - \mathbf{w}_x(\mathbf{u}_y\mathbf{v}_y + \mathbf{u}_z\mathbf{v}_z) \\
&= \mathbf{v}_x(\mathbf{u}_y\mathbf{w}_y + \mathbf{u}_z\mathbf{w}_z) - \mathbf{w}_x(\mathbf{u}_y\mathbf{v}_y + \mathbf{u}_z\mathbf{v}_z) + (\mathbf{u}_x\mathbf{v}_x\mathbf{w}_x - \mathbf{u}_x\mathbf{v}_x\mathbf{w}_x) \\
&= \mathbf{v}_x(\mathbf{u}_x\mathbf{w}_x + \mathbf{u}_y\mathbf{w}_y + \mathbf{u}_z\mathbf{w}_z) - \mathbf{w}_x(\mathbf{u}_x\mathbf{v}_x + \mathbf{u}_y\mathbf{v}_y + \mathbf{u}_z\mathbf{v}_z) \\
&= (\mathbf{u} \cdot \mathbf{w})\mathbf{v}_x - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}_x.
\end{aligned}$$

Similarly, the y and z components of $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ are given by:

$$\begin{aligned}
(\mathbf{u} \times (\mathbf{v} \times \mathbf{w}))_y &= (\mathbf{u} \cdot \mathbf{w})\mathbf{v}_y - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}_y \\
(\mathbf{u} \times (\mathbf{v} \times \mathbf{w}))_z &= (\mathbf{u} \cdot \mathbf{w})\mathbf{v}_z - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}_z
\end{aligned}.$$

By combining these three components we obtain:

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}.$$

□

Example 3.3. *Express* $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ *as a linear combination of* $\mathbf{u}$ *and* $\mathbf{v}$.

Proof. Since the cross product is anticommutative, this formula may also be written as:

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = -\mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = -(\mathbf{w} \cdot \mathbf{v})\mathbf{u} + (\mathbf{w} \cdot \mathbf{u})\mathbf{v}.$$

□

From Lagrange's formula it follows that the vector triple product satisfies:

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = 0$$

which is the Jacobi identity for the cross product. Another useful formula follows:

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) - \mathbf{v} \times (\mathbf{u} \times \mathbf{w}).$$

Example 3.4. *Let* $\alpha = (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c})$. *Show that*

$$\alpha = (\mathbf{c} \cdot \mathbf{a})|\mathbf{a}|^2 - ((\mathbf{c} \cdot \mathbf{a}))(\mathbf{a} \cdot \mathbf{b}).$$

Evaluate α *when* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$ *are unit vectors with* $\mathbf{b}$ *and* $\mathbf{c}$ *perpendicular, and the angles between* $\mathbf{a}$ *and* $\mathbf{b}$, *and between* $\mathbf{a}$ *and* $\mathbf{c}$, *are both* $\pi/3$.

Proof.

$$\begin{aligned}
(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c}) &= [\mathbf{a} \times \mathbf{b}, \mathbf{a}, \mathbf{c}] = [\mathbf{a}, \mathbf{c}, \mathbf{a} \times \mathbf{b}] = \mathbf{a} \cdot (\mathbf{c} \times (\mathbf{a} \times \mathbf{b})) \\
&= \mathbf{a} \cdot ((\mathbf{c} \cdot \mathbf{b})\mathbf{a} - (\mathbf{c} \cdot \mathbf{a})\mathbf{b}) = (\mathbf{c} \cdot \mathbf{b})(\mathbf{a} \cdot \ \mathbf{a}) - (\mathbf{c} \cdot \mathbf{a})(\mathbf{a} \cdot \mathbf{b}) \\
&= (\mathbf{c} \cdot \mathbf{a})|\mathbf{a}|^2 - ((\mathbf{c} \cdot \mathbf{a}))(\mathbf{a} \cdot \mathbf{b}).
\end{aligned}$$

With the given conditions, we have $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 1$, $\mathbf{c} \cdot \mathbf{b} = 0$, $\mathbf{a} \cdot \mathbf{b} = \mathbf{c} \cdot \mathbf{a} = 1 \times 1 \times \cos(\frac{\pi}{3}) = \frac{1}{2}$ and so

$$\alpha = 0 - (\frac{1}{2})^2 = \frac{1}{4}.$$

□