# 1 Introduction

In this chapter, two important differential equations viz. Legendre differential equation and Bessel's differential equation are studied. Properties of Legendre polynomial and Bessel's function are discussed. These functions has application in mathematical physics and engineering. Here, the definition of these functions, their main properties and some applications are presented.

# 2 Legendre Equation

Legendre polynomial is used in many different mathematical and physical problems. These polynomials may be constructed as a consequence of demanding a complete, orthogonal set of functions over the interval $[-1, 1]$. In quantum mechanics they represent angular momentum eigenfunctions.

## 2.1 Legendre Differential Equation

The equation

$$(1-z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + n(n+1)w = 0 \tag{1}$$

where $n$ is a real parameter, is known as Legendre equation of order $n$. This equation has regular singularities at $z = \pm 1$. The physically interesting range for $|z|$ is generally $|z| \leq 1$ (usually $z$ is the cosine of some angle). The equation (1) can be written as

$$\frac{d}{dz}\left[(1-z^2)\frac{dw}{dz}\right] + n(n+1)w = 0 \tag{2}$$

To check the singularity at infinity, we transform the equation by substituting $z = 1/t$. The transformed equation is

$$(t^2-1)\frac{d^2w}{dt^2} + 2t\frac{dw}{dt} + \frac{1}{t^2}n(n+1)w = 0 \tag{3}$$

which shows that the point $\infty$ is a regular singularity.

Further, this equation is a particular type of hypergeometric equation. This is done by substituting $z_1 = (1-z)/2$ which transforms $z = 1$ to $z_1 = 0$ and $z = -1$ to $z_1 = 1$. Also, $2dz_1 = -dz$

Under this transformation the Legendre equation becomes

$$\left(\frac{1-z}{2}\right)\left(\frac{1+z}{2}\right)\frac{d^2w}{dz_1^2} + \left[1 - (\overline{-n+n+1}+1)z_1\right]\frac{dw}{dz_1} + n(n+1)w = 0$$

$$\text{or} \quad z_1(1-z_1)\frac{d^2w}{dz_1^2} + \left[1 - (\overline{-n+n+1}+1)z_1\right]\frac{dw}{dz_1} - (-n)(n+1)w = 0$$

Here $a = -n$, $b = n + 1$, $c = 1$ and $z = z_1$. Therefore the solution of this equation is

$$w = F(-n, n + 1, 1; z_1) = F(-n, n + 1, 1; (1 - z)/2)$$

which shows that when $n$ is integer it reduces to a polynomial of degree $n$ and this is known as *Legendre function of first kind.*

## 2.2 Solution of Legendre Equation at $z = 0$

It is obvious that $z = 0$ is the ordinary point of the Legendre equation.

Let

$$w = \sum_{k=0}^{\infty} c_k z^k$$

be the solution of (1). Using this substitution (1) becomes

$$(1 - z^2)\sum_{k=0}^{\infty} c_k k(k-1)z^{k-2} - 2z\sum_{k=0}^{\infty} c_k k z^{k-1} + n(n+1)\sum_{k=0}^{\infty} c_k z^k = 0$$

The coefficient of $z^k$ leads to the following equation

$$c_{k+2}(k+1)(k+2) - c_k(k-1)k - 2c_k k + n(n+1)c_k = 0$$

and this equation can be written as

$$c_{k+2} = \frac{n(n+1) - k(k+1)}{(k+1)(k+2)}c_k = -\frac{(n-k)(n+k+1)}{(k+1)(k+2)}c_k$$

Thus

$$c_2 = -\frac{n(n+1)}{2!}c_0, \qquad c_3 = -\frac{(n-1)(n+2)}{2.3}c_1$$
$$c_4 = -\frac{(n-1)(n+3)}{3.4}c_2, \qquad c_5 = -\frac{(n-3)(n+4)}{4.5}c_3$$

and so on.

Thus the solution of (1) at $z = 0$ is

$$w = c_0\left[1 - \frac{n(n+1)}{2!}z^2 + \frac{n(n+1)(n-2)(n+3)}{4!}z^4 - \cdots\right]$$
$$+c_1\left[z - \frac{(n-1)(n+2)}{3!}z^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} - \cdots\right]$$

From this solution, it may be noted that whether $n$ be even or odd one of the series terminates. When $n$ is even, the first one terminates while the second gives an infinite series and otherwise. When $n$ is odd, first one is infinite and the second expression is a polynomial. The series which terminates is called Legendre polynomial of degree $n$ and is denoted by $P_n(z)$. The interval of convergence of the infinite series is $|z| < 1$. Moreover, $|z| \geq 1$ gives divergent series. Hence the general solution of Legendre equation is

$$w = AP_n(z) + BQ_n(z)$$

where $Q_n(z)$ is an infinite series and satisfies (1).

### 2.2.1 General expression of $P_n(z)$

Irrespective of the value of $n$(odd or even) the polynomial $P_n(z)$ of degree $n$ can be written as

$$P_n(z) = c_n\left[z^n + \frac{c_{n-2}}{c_n}z^{n-2} + \frac{c_{n-2}}{c_n}\frac{c_{n-4}}{c_{n-2}}z^{n-4} + \cdots\right]$$

where

$$c_{k+2} = -\frac{(n-k)(n+k+1)}{(k+1)(k+2)}c_k \quad k = n-2, n-4, \ldots$$

Thus

$$P_n(z) = c_n\left[z^n - \frac{n(n-1)}{2(2n-1)}z^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}z^{n-4} + \cdots p \text{ terms}\right]$$

where $p = n/2 + 1$, when $n$ is even and $p = (n-1)/2 + 1$, when $n$ is odd.

Since the constant $c_n$ is arbitrary, we can choose $c_n = \dfrac{a.3.5\ldots(2n-1)}{n!}$ and hence the Legendre polynomial becomes

$$P_n(z) = \frac{1.3.5.\ldots.(2n-1)}{n!}\left[z^n - \frac{n(n-1)}{2(2n-1)}z^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}z^{n-4} + \cdots + p \text{ terms}\right]$$

$$= \frac{(2n)!}{2^n(n!)^2}\left[z^n - \frac{n(n-1)}{2(2n-1)}z^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}z^{n-4} + \cdots + p \text{ terms}\right]$$

The last term is constant when $n$ is even or contain $z$ when $n$ is odd.

Some lower order Legendre polynomials are

$$P_0(z) = 1$$
$$P_1(z) = z$$
$$P_2(z) = \frac{1}{2}(3z^2 - 1)$$
$$P_3(z) = \frac{1}{2}(5z^3 - 3z)$$
$$P_4(z) = \frac{1}{8}(35z^4 - 30z^2 + 3)$$
$$P_5(z) = \frac{1}{8}(63z^5 - 70z^3 + 15z)$$
$$P_6(z) = \frac{1}{16}(231z^6 - 315z^4 + 105z^2 - 5)$$
$$P_6(z) = \frac{1}{16}(429z^7 - 693z^4 + 315z^3 - 35z)$$
$$P_7(z) = \frac{1}{128}(6435z^8 - 1201z^6 + 6930z^4 - 1260z^2 + 35)$$

Conversely the terms 1, $z$, $z^2$ etc. can be expressed using Legendre polynomial. Thus every algebraic function is expressible in terms of Legendre polynomial

$$1 = P_0(z)$$
$$z = P_1(z)$$
$$z^2 = \frac{1}{3}[2P_2(z) + 1]$$
$$z^3 = \frac{1}{5}[2P_3(z) + 3P_1(z)]$$
$$z^4 = \frac{1}{35}[8P_4(z) + 20P_2(z) + 7]$$
$$z^5 = \frac{1}{63}[8P_5(z) + 28P_3(z) + 27P_1(z)] \text{ and so on.}$$

**Example 2.1** Show that

$$4z^3 - 5z^2 - 3z = \frac{8}{5}P_3(z) - \frac{10}{3}P_2(z) - \frac{3}{5}P_1(z) - \frac{5}{3}P_0(z)$$

**Solution.** $4z^3 - 5z^2 - 3z = \frac{4}{5}[2P_3(z) + 3P_1(z)] - \frac{5}{3}[2P_2(z) + 1] - 3P_1(z)$

$$= \tfrac{8}{5}P_3(z) - \tfrac{10}{3}P_2(z) - \tfrac{3}{5}P_1(z) - \tfrac{5}{3}P_0(z)$$

## 2.3 Rodrigues Formula

The Legendre polynomial is

$$P_n(z) = \frac{(2n)!}{2^n(n!)^2}\left[z^n - \frac{n(n-1)}{2(2n-1)}z^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}z^{n-4} + \cdots + p \text{ terms}\right]$$

The last term being a constant or multiple of $z$ according as $m$ is even or odd. When $n$ is even number of term is $\frac{n}{2} + 1$ and when $n$ is odd number of term is $\frac{n-1}{2} + 1$. So the number of term is denoted by $p$.

$$P_n(z) = \frac{1.3.5.\ldots.(2n-1)}{n!}\frac{d}{dz}\Big[\frac{z^{n+1}}{n+1} - \frac{n(n-1)}{2(2n-1)(n-1)}z^{n-1} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)(n-3)}z^{n-3} + \cdots + p \text{ terms}\Big]$$

$$= \frac{1.3.5.\ldots.(2n-1)}{(n+1)!}\frac{d}{dz}\big[z^{n+1} - \frac{(n+1)n}{2(2n-1)}z^{n-1} + \frac{(n+1)n(n-1)(n-2)}{2.4(2n-1)(2n-3)}z^{n-3} + \cdots + p \text{ terms}\big]$$

$$= \frac{1.3.5.\ldots.(2n-1)}{(n+2)!}\frac{d^2}{dz^2}\big[z^{n+2} - \frac{(n+2)(n+1)}{2(2n-1)}z^n + \frac{(n+2)(n+1)n(n-1)}{2.4(2n-1)(2n-3)}z^{n-2} + \cdots + p \text{ terms}\big]$$

$$= \frac{1.3.5.\ldots.(2n-1)}{(2n)!}\frac{d^n}{dz^n}\big[z^{2n} - \frac{2n(2n-1)}{2(2n-1)}z^{2n-2} + \frac{2n(2n-1)(2n-2)(2n-3)}{2.4(2n-1)(2n-3)}z^{2n-4} + \cdots + p \text{ terms}\big]$$

The last term being a multiple of $z^n$ or $z^{n+1}$ according as $n$ is even or odd. Therefore

$$P_n(z) = \frac{1.3.5.\ldots.(2n-1)}{(2n)!}\frac{d^n}{dz^n}\big[z^{2n} - \frac{2n(2n-1)}{2(2n-1)}z^{2n-2} + \frac{2n(2n-1)(2n-2)(2n-3)}{2.4(2n-1)(2n-3)}z^{2n-4} + \cdots + (n+1) \text{ terms}\big]$$

The terms that are added within the third bracket to make $(n+1)$ terms are at most of degree $z^{n-2}$ when $n$ is even and $z^{n-1}$ when $n$ is odd. So that these terms when differentiated $n$ times with respect to $z$ become zero. Therefore,

$$\begin{aligned} P_n(z) &= \frac{1.3.5.\ldots.(2n-1)}{(2n)!}\frac{d^n}{dz^n}\big[(z^2-1)^n\big] \\ &= \frac{1}{2.4.6.\ldots.(2n)}\frac{d^n}{dz^n}\big[(z^2-1)^n\big] \\ &= \frac{1}{2^n n!}\frac{d^n}{dz^n}\big[(z^2-1)^n\big] \end{aligned} \tag{4}$$

The equation (4) is called Rodrigues formula for Legendre polynomial.

The Legendre polynomial has many interesting properties, some of them can be deduced from Rodrigues formula.

Recall that $P_n(z)$ is a solution of Legendre equation

$$(1-z^2)w'' - 2zw' + n(n+1)w = 0$$

and since the equation is unchanged on changing $n$ to $-n-1$, therefore

$$P_n(z) = P_{-n-1}(z)$$

The Legendre polynomial is even or odd function according as $n$ is even or odd, i.e.,

$$P_n(-z) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dz^n}\big[(z^2-1)^n\big] = (-1)^n P_n(z) \tag{5}$$

The lower order polynomial can be deduced from Rodrigues formula by substituting
$n = 0, 1, 2, \ldots$.

The value of the Legendre polynomial at $z = 1$ is unity, i.e.,

$$P_n(1) = 1$$

This proposition is justified in the following.

$$P_n(1) = \frac{1}{2^n n!} \frac{d^n}{dz^n} \left[(z+1)^n (z-1)^n\right]_{z=1}$$

Differentiating $\left[(z+1)^n (z-1)^n\right]$, $n$ times using Leibnitz theorem

$$\begin{aligned} P_n(1) &= \frac{1}{2^n n!} \left[n!(z+1)^n + n(n-1)!(z+1)^{n-1}(z-1) + \cdots + n!(z-1)^n\right]_{z=1} \\ &= \frac{1}{2^n n!} \left[n! 2^n + 0\right] = 1 \end{aligned}$$

**Example 2.2** Use Rodrigues formula to show that

$$P'_{n+1}(z) - P'n - 1(z) = (2n+1)P_n(z),$$

where prime represents the derivative with respect to $z$

**Solution.**

$$\begin{aligned} & P'_{n+1}(z) - P'n - 1(z) \\ &= \frac{d}{dz}\left[\frac{1}{2^{n+1}(n+1)!} \frac{d^{n+1}}{dz^{n+1}} \left[(z^2-1)^{n+1}\right]\right] + \frac{d}{dz}\left[\frac{1}{2^{n-1}(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[(z^2-1)^{n-1}\right]\right] \\ &= \frac{1}{2^{n+1}(n+1)!} \frac{d^{n+2}}{dz^{n+2}} \left[(z^2-1)^{n+1}\right] + \frac{1}{2^{n-1}(n-1)!} \frac{d^n}{dz^n} \left[(z^2-1)^{n-1}\right] \\ &= \frac{1}{2^n n!} \frac{d^n}{dz^n}\left[\frac{1}{2(n+1)} \frac{d^2}{dz^2} \left[(z^2-1)^{n+1}\right] - 2n(z^2-1)^{n-1}\right] \\ &= \frac{1}{2^n n!} \frac{d^n}{dz^n}\left[\frac{d}{dz} \left[z(z^2-1)^n\right] - 2n(z^2-1)^{n-1}\right] \\ &= \frac{1}{2^n n!} \frac{d^n}{dz^n} \left[(z^2-1)^n + 2n(z^2-1)^{n-1} - 2n(z^2-1)^{n-1}\right] \\ &= \frac{1}{2^n n!} \frac{d^n}{dz^n} \left[(z^2-1)^n + 2n(z^2-1)(z^2-1)^{n-1}\right] \\ &= \frac{1}{2^n n!} \frac{d^n}{dz^n} \left[(2n+1)(z^2-1)^n\right] \\ &= \frac{(2n+1)}{2^n n!} \frac{d^n}{dz^n} \left[(z^2-1)^n\right] \\ &= (2n+1)P_n(z) \end{aligned}$$

The Legendre polynomial of order $n$ contains $n$ zeros and these zeros lie between $-1$ and 1. This proposition is proved in the following theorem.

**Theorem 1** *The zeros of Legendre polynomial are all real, distinct and lie between* $-1$ *and* $1$.

**Proof:** Let us consider the Rodrigues formula

$$P_n(z) = \frac{1}{2^n n!}\frac{d^n}{dz^n}\left[(z^2-1)^n\right]$$

Rewriting this equation becomes

$$P_n(z) = \frac{1}{2^n n!}\frac{d^n}{dz^n}\left[(z+1)^n(z-1)^n\right]$$

and applying Leibnitz theorem $n$ times we obtain

$$P_n(z) = \frac{1}{2^n n!}\left[n!(z+1)^n + n(n-1)!(z-1)(z+1)^{n-1} + \cdots + n!(z-1)^n\right]$$

If $z$ be real and $|z| > 1$ all the terms in the above are positive, i.e., $P_n(z)$ is always positive. Hence there cannot be any zero for all $z$ with $|z| > 1$. Hence the zeros lie between $-1$ and $1$.

Again, the equation $(z^2-1)^n = 0$ has $2n$ roots.

Among these roots $n$ roots are positive (1) and $n$ roots are negative (-1).

Therefore by Rolle's theorem, the equation $\frac{d}{dz}\left[(z^2-1)^n\right] = 0$ must have at least one root between $-1$ and $1$ also this is the same as

$$(z+1)^{n-1}n(z-1)^n + n(z+1)^n(z-1)^{n-1} = 0$$
$$\text{or} \quad (z+1)^{n-1}(z-1)^{n-1}.2z = 0$$

Hence the equation $\frac{d}{dz}\left[(z^2-1)^n\right] = 0$ has $(2n-1)$ roots, $n-1$ roots equal to $1$ and $n-1$ roots equal to $-1$ and one root, say $\alpha_0$ $(=0)$ between $-1$ and $1$.

Again, from the above result it follows that the equation $\frac{d^2}{dz^2}\left[(z^2-1)^n\right] = 0$ must have at least one root, say $\alpha_1$ (say) between $-1$ and $\alpha_0$ and at least one root $\alpha_2$ (say) between $\alpha_0$ and $1$. Hence $\frac{d^2}{dz^2}\left[(z^2-1)^n\right] = 0$ has two roots between $-1$ and $1$.

But,

$$\frac{d^2}{dz^2}\left[(z^2-1)^n\right] = (z-1)^{n-2}(z+1)^{n+2}f_2(z) = 0$$

where $f_2(z)$ is a polynomial of degree 2 of $z$. Thus the equation has $n-2$ roots equal to $1$, $n-2$ roots equal to $-1$ and two roots $\alpha_1$ and $\alpha_2$ between $-1$ and $1$.

Similarly, $\frac{d^3}{dz^3}\left[(z^2-1)^n\right] = 0$ has three roots $\alpha_1'$, $\alpha_2'$, $\alpha_3'$ between $-1$ and $\alpha_1$; $\alpha_1$ and $\alpha_2$; $\alpha_2$ and $1$.

Proceeding this way, $\frac{d^n}{dz^n}\left[(z^2-1)^n\right] = 0$ has $n$ roots all real, distinct and lie between $-1$ and $1$. Hence the theorem follws.

## 2.4 Orthogonality

Let $z$ be real and $P_m(z)$ and $P_n(z)$ are the Legendre polynomial of degree $m$ and $n$ respectively. Since the Legendre polynomial is the solution of Legendre equation, therefore

$$\frac{d}{dz}\left[(1-z^2)\frac{dP_m}{dz}\right] + m(m+1)P_m = 0 \tag{6}$$

and

$$\frac{d}{dz}\left[(1-z^2)\frac{dP_n}{dz}\right] + n(n+1)P_n = 0 \tag{7}$$

Multiplying (6) by $P_n$ and (7) by $P_m$ and subtracting we have

$$\frac{d}{dz}\left[(1-z^2)(P'_mP_n - P'_nP_m)\right] + [m(m+1) - n(n+1)]P_mP_n = 0$$

Integrating between the limits $-1$ and 1 with respect to $z$

$$\left[(1-z^2)(P'_mP_n - P'_nP_m)\right]_{-1}^{1} + \int_{-1}^{1}[m(m+1) - n(n+1)]P_mP_n dz = 0$$

$$\text{or} \quad [m(m+1) - n(n+1)]\int_{-1}^{1} P_mP_n dz = 0$$

$$\text{or} \quad \int_{-1}^{1} P_mP_n dz = 0 \quad \text{since} \quad m \neq n$$

It can be shown that

$$\int_{-1}^{1} P_n(z)^2 dz = \frac{2}{2n+1}$$

This fact is justified by considering the Rodrigues formula in the following.

$$\int_{-1}^{1} P_n(z)^2 dz = A\int_{-1}^{1} u_n^{(n)}u_n^{(n)} dz$$

where $A = \dfrac{1}{(2^n n!)^2}$ and $u_n = (z^2-1)^n$, $u_n^{(n)}$ denotes the $n$th derivative of

$u_n$

$$= \left[u_n^{(n-1)}u_n^{(n)}\right]_{-1}^{1} - A\int_{-1}^{1} u_n^{(n-1)}u_n^{(n+1)}dz$$

$$= 0 + (-1)A\int_{-1}^{1} u_n^{(n-1)}u_n^{(n+1)}dz$$

$$= (-1)^n A\int_{-1}^{1} u_n u_n^{(2n)}dz$$

$$= (-1)^n(2n)!A\int_{-1}^{1} u_n dz \quad \text{since} \quad u_n^{(2n)} = (2n)!$$

Now,

$$\int_{-1}^{1}(z^2-1)^n dz = \int_{-1}^{1}(z+1)^n(z-1)^n dz$$

Putting $1 - z = 2y \Rightarrow -dz = 2dy$

Therefore the integral $= (-1)^n \int_{1}^{0} 2^{2n+1}y^n(1-y)^n(-dz)$

$$= (-1)^n 2^{2n+1}B(n+1,n+1) = (-1)^n 2^{2n+1}\frac{(n!)^2}{(2n+1)!}$$

Hence

$$\int_{-1}^{1} P_n^2(z)dz = A(-1)^n(2n)!(-1)^n 2^{2n+1}\frac{(n!)^2}{(2n+1)!}$$

**Note:** The orthogonality of Legendre polynomial in terms of trigonometric form is

$$P_m(\cos\theta)P_n(\cos\theta)\sin\theta d\theta = \begin{cases} 0, & if\ m \neq n \\ \dfrac{2}{2n+1}, & if\ m = n \end{cases}$$

.

## 2.5 Expansion of Functions: Legendre Series

If $f(z)$ be a piecewise continuous function then it can be expanded in Legendre series. Let us assume that the series

$$\sum_{n=0}^{\infty} a_n P_n(z) = f(z) \tag{8}$$

in the sense of convergence in the interval $[-1, 1]$. This means that the function $f(z)$ and $f'(z)$ be at least peace wise continuous in this interval. To find $a_n$, multiplying (8) by $P_m(z)$, integrating term by term and then using orthogonal property, we obtain

$$\frac{2}{2m+1}a_m = \int_{-1}^{1} f(z)P_m(z)dz$$

Changing the variable of integration $z$ by $t$ and the index $m$ by $n$, we have

$$a_n = \frac{2n+1}{2}\int_{-1}^{1} f(t)P_n(t)dt \tag{9}$$

The constant $a_n$ are called Legendre constants.

Using (9), (8) becomes

$$f(z) = \sum_{n=0}^{\infty} \frac{2n+1}{2}\int_{-1}^{1} (f(t)P_n(t)dt)P_n(z) \tag{10}$$

This expansion in a series of Legendre polynomial is usually referred to as a Legendre series.

Legendre polynomial is also satisfy the following properties.

**Property 1** *Prove that*

$$\int_{-1}^{1} P_n(z)dz = \begin{cases} 0, & if\ n \neq 0 \\ 2, & if\ n = 0 \end{cases}$$

.

**Proof:** Since $P_n(z)$ is a solution of Legendre equation, then

$$\frac{d}{dz}\left[(z^2-1)\frac{dP_n}{dz}\right] + n(n+1)P_n = 0$$

Integrating between the limits $-1$ and 1, we get

$$\int_{-1}^{1} \frac{d}{dz}\left[(z^2-1)\frac{dP_n}{dz}\right]dz + \int_{-1}^{1} n(n+1)P_n(z)dz = 0$$

$$\text{or} \quad \left[(z^2-1)\frac{dP_n}{dz}\right]_{-1}^{1} + n(n+1)\int_{-1}^{1} P_n(z)dz = 0$$

$$\text{or} \quad 0 + n(n+1)\int_{-1}^{1} P_n(z)dz = 0$$

$$\text{or} \quad \int_{-1}^{1} P_n(z)dz = 0, \quad n \neq 0$$

When $n = 0$,

$$\int_{-1}^{1} P_n(z)dz = \int_{-1}^{1} P_0(z)dz = \int_{-1}^{1} dz = 2$$

**Property 2** *For integers $m$ and $n$*

$$\int_{-1}^{1} z^m P_n(z)dz = \begin{cases} 0, & if\ m < n \\ \dfrac{2^{n+1}(n!)^2}{(2n+1)!}, & if\ m = n \end{cases}$$

.

**Proof:** Let $A = \dfrac{1}{2^n n!}$ and $u_n = (z^2-1)^n$

Then

$$\int_{-1}^{1} z^m P_n(z)dz = A\int_{-1}^{1} z^m u_n^{(}n)dz$$

$$= A\left[u_n^{(n-1)}z^m\right]_{-1}^{1} - A\int_{-1}^{1} mz^{m-1}u_n^{(n-1)}dz$$

$$= (-1)A\int_{-1}^{1} mz^{m-1}u_n^{(n-1)}dz$$

$$= (-1)^2 Am\int_{-1}^{1}(m-1)z^{m-2}u_n^{(n-2)}dz$$

$$= (-1)^m Am(m-1)(m-2)\dots 2.1\int_{-1}^{1} u_n^{(n-m)}dz \tag{11}$$

$$= (-1)^m Am!\left[u_n^{(n-m-1)}\right]_{-1}^{1}$$

$$= (-1)^m Am!0 = 0, \quad \text{since} \quad n > m, n-m-1 \geq 0, \quad \text{so } u_n^{(n-m-1)} \text{ contains at least one } z^2-1.$$

From equation (11), for $n = m$ we have

$$\int_{-1}^{1} z^m P_n(z)dz = (-1)^n An!\int_{-1}^{1} u_n(z)dz$$

$$= (-1)^n \frac{1}{2^n n!}n!(-1)^n\frac{(n!)^2}{(2n)!}\frac{2^{2n+1}}{2n+1} = \frac{2^{n+1}(n!)^2}{(2n+1)!}$$

## 2.6 Generating Function

The generating function for Legendre polynomial is

$$g(z,t) = (1-2zt+t^2)^{-\frac{1}{2}}, \quad |\, z\,| \leq 1, \quad |\, t\,| < 1$$

That is, the coefficient of $t^n$ is the Legendre polynomial of order $n$ is $(1-2zt+t^2)^{-\frac{1}{2}}$. This generating function generates the Legendre polynomial of all orders.

For $|\, z\,| \leq 1$ and $|\, t\,| < 1$, the expansion of $(1-2zt+t^2)^{-\frac{1}{2}}$ is shown below.

Since $|\, z\,| \leq 1$ and $|\, t\,| < 1$, $(1-2zt+t^2)^{-\frac{1}{2}}$ can be expanded in power series.

$$(1-2zt+t^2)^{-\frac{1}{2}} = [1-t(2z-t)]^{-\frac{1}{2}}$$

$$= 1+\frac{1}{2}t(2z-t)+\frac{1.3}{2.4}t^2(2z-t)^2+\dots+\frac{1.3.\dots.(2n-1)}{2.4\dots 2n}t^n(2z-t)^n+\cdots$$

Now the coefficient of $t^n$ in the term

$$\frac{1.3.\ldots.(2n-1)}{2.4\ldots 2n}t^n(2z-t)^n \text{ is } \frac{1.3.\ldots.(2n-1)}{2.4\ldots 2n}(2z)^n$$

$$= \frac{1.3.\ldots.(2n-1)}{2.4\ldots 2n}2^n z^n$$

$$= \frac{1.3.\ldots.(2n-1)}{n!}z^n$$

Again the coefficient of $t^n$ in the term

$$\frac{1.3.\ldots.(2n-3)}{2.4\ldots(2n-2)}t^{n-1}(2z-t)^{n-1} \text{ is } \frac{1.3.\ldots.(2n-3)}{2.4\ldots(2n-2)}\{-(n-1)(2z)^{n-2}\}$$

$$= -\frac{1.3.\ldots.(2n-3)}{2^{n-1}1.2\ldots(n-1)}\frac{2n-1}{n}\frac{n}{2n-1}\{(n-1)(2)^{n-2}z^{n-2}\}$$

$$= -\frac{1.3.\ldots.(2n-2)}{n!}\frac{n(n-1)}{2(2n-1)}z^{n-2}$$

and so on.

Hence the coefficient of $t^n$ in the expression of $(1-2zt+t^2)^{-\frac{1}{2}}$ is

$$\frac{1.3.\ldots.(2n-2)}{n!}\left[z^n - \frac{n(n-1)}{2(2n-1)}z^{n-2} + \cdots\right]$$

i.e., the coefficient of $t^n$ in the expression of $(1-2zt+t^2)^{-\frac{1}{2}}$ is $P_n(z)$.

$$\text{Thus } (1-2zt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(z) \tag{12}$$

## 2.7 Special values

The generating function gives more information about Legendre polynomials.

For $z=1$ the equation (12) becomes

$$(1-2t+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(1)$$

$$\text{or } \frac{1}{1-t} = \sum_{n=0}^{\infty} t^n P_n(1)$$

$$\text{or } \sum_{n=0}^{\infty} t^n = \sum_{n=0}^{\infty} t^n P_n(1)$$

Hence $P_n(1)=1$ for all values of $n$.

For $z=0$ the equation (12) becomes

$$(1+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(z)t^n$$

Also, by binomial expansion

$$(1+t^2)^{-\frac{1}{2}} = 1 - \frac{1}{2}t^2 + \frac{3}{8}t^4 + \cdots + (-1)^n \frac{1.3.\ldots.(2n-1)}{2^n n!}t^{2n} + \cdots$$

Hence $P_{2n}(0)$ =coefficient of $t^{2n}$ in the expansion of $(1+t^2)^{-\frac{1}{2}}$

$$= (-1)^n \frac{1.3.\ldots.(2n-1)}{2^n n!} = \frac{(-1)^n (2n)!}{2^{2n}(n!)^2}$$

$P_{2n+1}(0)$ =coefficient of $t^{2n+1}$ in the expansion of $(1+t^2)^{-\frac{1}{2}}$ is 0.

The generating function remains unchanged, if we replace $z$ by $-z$ and $t$ by $-t$.

That is,

$$g(t,z) = g(-t,-z) = [1 - 2(-t)(-z) + (-z)^2]^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(-z)(-t)^n$$

i.e.,

$$\sum_{n=0}^{\infty} P_n(z)(t)^n = \sum_{n=0}^{\infty} (-1)^n P_n(-z)(t)^n$$

Hence

$$P_n(-z) = (-1)^n P_n(z) \tag{13}$$

That is, the Legendre polynomials are odd or even with respect to $z = 0$ according to whether the index is odd or even. This is called parity or reflection property of Legendre polynomial.

From the relation (13), it is easy to observe that

$$P_{2n+1} = 0$$

The Legendre polynomial can be expressed in terms of cosine function. To find the expression for $P_n(\cos\theta)$, recall the generating function (12).

Substituting $z = \cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ to

$$(1 - 2zt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(z)(t)^n$$

we get

$$\sum_{n=0}^{\infty} P_n(\cos\theta)(t)^n = \left[1 - t(e^{i\theta} + e^{-i\theta}) + t^2\right]^{-\frac{1}{2}} = (1 - te^{i\theta})^{-\frac{1}{2}}(1 - te^{-i\theta})^{-\frac{1}{2}}$$

Using binomial expansion, we have

$$(1 - te^{i\theta})^{-\frac{1}{2}} = 1 + \frac{1}{2}te^{i\theta} + \frac{1.3}{2.4}t^2 e^{2i\theta} + \cdots + \frac{1.3.\ldots.(2n-1)}{2.4.\ldots.(2n)}t^n e^{in\theta} + \cdots$$

$$(1-te^{-i\theta})^{-\frac{1}{2}} = 1+\frac{1}{2}te^{-i\theta}+\frac{1.3}{2.4}t^2e^{-2i\theta}+\cdots+\frac{1.3.\ldots.(2n-1)}{2.4.\ldots.(2n)}t^ne^{-in\theta}+\cdots$$

Multiplying the above two expression and equating the coefficient of $t^n$ we obtain

$$P_n(\cos\theta) = \frac{1.3.\ldots.(2n-1)}{2.4.\ldots.(2n)}2\cos n\theta + \frac{1}{2}\frac{1.3.\ldots.(2n-3)}{2.4.\ldots.(2n-2)}2\cos(n-2)\theta + \quad (14)$$

**Example 2.3** If $(R,\phi,z)$ and $(r,\theta,\phi)$ are cylindrical polar and spherical polar coordinates of the same point, show that

$$P_n(\cos\theta) = (-1)^n\frac{r^{n+1}}{n!}\frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right)$$

**Solution.** We have $r^2 = x^2+y^2+z^2 = R^2+z^2$. So $\dfrac{1}{r} = (R^2+z^2)^{-\frac{1}{2}} = f(R,z)$.

Then $f(R,z+h) = \left[R^2+(z+h)^2\right]^{-\frac{1}{2}} = \left[R^2+z^2+2zh+h^2\right]^{-\frac{1}{2}}$

$$= [r^2+2hr\cos\theta+h^2]^{-\frac{1}{2}} = \frac{1}{r}\left[1+\frac{2h\cos\theta}{r}+\frac{h^2}{r^2}\right]^{-\frac{1}{2}}$$

$$= \frac{1}{r}\sum_{n=0}^{\infty}(-1)^n\left(\frac{h}{r}\right)^n P_n(\cos\theta) \quad (15)$$

Again, by Taylor's series, for several variables

$$f(R,z+h) = \sum_{n=0}^{\infty}\frac{h^n}{n!}\frac{\partial^n}{\partial z^n}f(R,z) = \sum_{n=0}^{\infty}\frac{h^n}{n!}\frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right) \quad (16)$$

From the equation (15) and (16) we have

$$\frac{1}{r}\sum_{n=0}^{\infty}(-1)^n\left(\frac{h}{r}\right)^n P_n(\cos\theta) = \sum_{n=0}^{\infty}\frac{h^n}{n!}\frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right)$$

Equating the coefficient of $h^n$ on both sides of the above equation, we get

$$\frac{(-1)^n}{r^{n+1}}P_n(\cos\theta) = \frac{1}{n!}\frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right)$$

That is,

$$P_n(\cos\theta) = \frac{(-1)^n}{n!}r^{n+1}\frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right)$$

## 2.8 Bounds of $P_n(\cos\theta)$

The generating function may be used to find the bounds of $P_n(\cos\theta)$.

We have for $z = \cos\theta$

$$\left[1-t(e^{i\theta}+e^{-i\theta})+t^2\right]^{-\frac{1}{2}} = (1-te^{i\theta})^{-\frac{1}{2}}(1-te^{-i\theta})^{-\frac{1}{2}}$$

$$= \left(1 + \frac{1}{2}te^{i\theta} + \frac{1.3}{2.4}t^2e^{2i\theta} + \cdots\right) \times \left(1 + \frac{1}{2}te^{-i\theta} + \frac{1.3}{2.4}t^2e^{-2i\theta} + \cdots\right) \quad (17)$$

It may be noted that all the coefficient of above expression are positive. So the each term of right hand side of (17) is positive. Each term of the (17) is of the form

$$a_m \frac{(e^{im\theta} + e^{-im\theta})}{2} = a_m \cos m\theta$$

where $a_m$ is positive. Then (17) reduces to

$$P_n(\cos\theta) = \sum_{m=0 \text{ or } 1}^{n} a_m \cos m\theta \quad (18)$$

The right hand side of (18) is clearly maximum when $\theta = 0$ and $m\theta = 1$. That is, the maximum value of $P_n(\cos\theta)$ is $P_n(1)$. But for $z = \cos\theta = 1$, $P_n(1) = 1$.

Therefore $|P_n(\cos\theta)| \leq P_n(1) = 1$ i.e., $-1 \leq P_n(\cos\theta) \leq 1$

From the equation (14) and(18) we may conclude that the Legendre polynomial is a linear combination of $\cos m\theta$. This means that the Legendre polynomials form a complete set for any functions that may be expanded by a Fourier cosine series over the interval $(0, \pi)$.

## 2.9 Recurrence Relations

The generating function for Legendre polynomials provides a better way to deduce the recurrence relations and some special properties. Differentiating the relation

$$(1 - 2zt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(z) \quad (19)$$

with respect to $t$ we obtain

$$-\frac{1}{2}(1 - 2zt + t^2)^{-\frac{3}{2}}(-2z + 2t) = \sum_{n=0}^{\infty} nt^{n-1}P_n(z)$$

$$\text{or} \quad (z - t)(1 - 2zt + t^2)^{-\frac{1}{2}} = (1 - 2zt + t^2)\sum_{n=0}^{\infty} nt^{n-1}P_n(z)$$

$$\text{or} \quad (z - t)\sum_{n=0}^{\infty} t^n P_n(z) = (1 - 2zt + t^2)\sum_{n=0}^{\infty} nt^{n-1}P_n(z)$$

Equating the coefficient of $t^n$ from both sides, we get

$$zP_n(z) - P_{n-1}(z) = (n + 1)P_{n+1}(z) + nP_{n-1}(z)$$

On simplification,

$$(2n+1)zP_n(z) = (n+1)(n+1)P_{n+1}(z) + nP_{n-1}(z) \tag{20}$$

Again differentiating the relation (19) with respect to $t$ we obtain

$$-\frac{1}{2}(1-2zt+t^2)^{-\frac{3}{2}}(-2z+2t) = \sum_{n=0}^{\infty} nt^{n-1}P_n(z) \tag{21}$$

Differentiating the relation (19) with respect to $z$ we obtain

$$-\frac{1}{2}(1-2zt+t^2)^{-\frac{3}{2}}(-2t) = \sum_{n=0}^{\infty} t^n P_n'(z) \tag{22}$$

Dividing (21) by (22)

$$\frac{t-z}{-t} = \frac{\sum_{n=0}^{\infty} nt^{n-1}P_n(z)}{\sum_{n=0}^{\infty} t^n P_n'(z)}$$

That is,

$$\sum_{n=0}^{\infty} t^{n+1}P_n'(z) - z\sum_{n=0}^{\infty} t^n P_n'(z) + \sum_{n=0}^{\infty} nt^n P_n(z)$$

Equating coefficient of $t^n$ to zero,

$$P_{n-1}'(z) - zP_n'(z) + nP_n(z) = 0$$

On simplification we get

$$nP_n(z) = zP_n'(z) - P_{n-1}'(z) \tag{23}$$

Another recurrence relation can be obtained by differentiating the relation (19) with respect to $z$. Differentiating the relation (19) with respect to $z$ we obtain

$$-\frac{1}{2}(1-2zt+t^2)^{-\frac{3}{2}}(-2t) = \sum_{n=0}^{\infty} t^n P_n'(z)$$

Differentiating the relation (19) with respect to $z$ we obtain

$$t\sum_{n=0}^{\infty} t^n P_n(z) = (1-2zt+t^2)\sum_{n=0}^{\infty} t^n P_n'(z)$$

Equating the coefficient of $t^{n+1}$ in both sides we obtain

$$P_n(z) = zP_{n+1}'(z) - 2zP_n'(z) + P_{n-1}'(z) \tag{24}$$

From (23),

$$zP'_n(z) = nP_n(z) + P'_{n-1}(z)$$

Substituting this value in (24), we get

$$P_n(z) = zP'_{n+1}(z) - 2[nP_n(z) + P'_{n-1}(z)] + P'_{n-1}(z)$$

$$\text{or} \quad (2n+1)P_n(z) = P'_{n+1}(z) - P'_{n-1}(z) \tag{25}$$

Combination of (23) and (24) yields

$$P_n(z) = P'_{n+1}(z) - 2zP'_n(z) + [zP'_n(z) - nP_n(z)]$$

$$\text{i.e.,} \quad (n+1)P_n(z) = P'_{n+1}(z) - zP'_n(z) \tag{26}$$

Replacing $n$ by $n-1$ we get

$$nP_{n-1}(z) = P'_n(z) - zP'_{n-1}(z) \tag{27}$$

Multiplying (26) by $z$ and adding with (27) we get

$$\begin{aligned} &nP_{n-1}(z) + z(n+1)P_n(z) = zP'_{n+1}(z) - z^2P'_n(z) + P'_n(z) - zP'_{n-1}(z) \\ &= (1-z^2)P'_n(z) - z(P'_{n+1}(z) - P'_{n-1}(z)) \\ &= (1-z^2)P'_n(z) + z(2n+1)P_n(z) \quad [\text{Using} \quad (25)] \end{aligned}$$

This relation can be written as

$$(1-z^2)P'_n(z) = n[(P_{n-1}(z) - zP_n(z)] \tag{28}$$

Differentiating (28) we have

$$\begin{aligned} &(1-z^2)P''_n(z) - 2zP'_n(z) = nP'_{n-1}(z) - nP_n(z) - nzP'_n(z) \\ &= n[zP'_n(z) - nP_n(z)] - nP_n(z) - nzP'_n(z) \quad [\text{Using (23)}] \end{aligned}$$

Thus

$$(1-z^2)P''_n(z) - 2zP'_n(z) + n(n+1)P_n(z) = 0 \tag{29}$$

The equation (20), (23), (25), (26) and (28) are the recurrence relations for Legendre polynomials.

The equations (23), (25), (26) and (28) are all first order differential equations but with two different indices. the indices of (29) are equal but of second order differential equation and this equation is Legendre equation.

It may be noted that the recurrence relations are deduced from the generating function of the Legendre polynomial without considering the Legendre differential equation and finally we obtain the Legendre differential equation (29).

The equation (29) can be written as

$$\frac{d}{dz}\left[(1-z^2)\frac{dP_n(z)}{dz}\right]+n(n+1)P_n(z)=0$$

Substituting $z=\cos\theta$, $\quad \frac{dz}{d\theta}=-\sin\theta$

$$\frac{d}{d\theta}\left[(1-z^2)\frac{dP_n(z)}{d\theta}\frac{d\theta}{dz}\right]+n(n+1)P_n(z)=0$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{dP_n(\cos\theta)}{d\theta}\right]+n(n+1)P_n(\cos\theta)=0 \tag{30}$$

## 2.10 Integral Representation

**Schlafli's Integral**

The Rodrigues formula is

$$P_n(z)=\frac{1}{2^n n!}\frac{d^n}{dz^n}\{(z^2-1)^n\}$$

From the theory of analytic function, it is known that if $f(z)$ be analytic than its $n$th derivative can be represented in the integral form

$$\frac{d^n}{dz^n}\{f(z)\}=\frac{n!}{2\pi i}\int_C\frac{f(t)}{(t-z)^{n+1}}dt$$

where $C$ is a closed contour within and on which $f(z)$ is analytic.

The function $(z^2-1)^n$ is analytic for all values of $z$. Therefore

$$\frac{d^n}{dz^n}\{(z^2-1)^n\}=\frac{n!}{2\pi i}\int_{C_0}\frac{(t^2-1)^n}{(t-z)^{n+1}}dt$$

where $C_0$ is a closed contour in $t$-plane enclosing $t=z$.

Therefore

$$P_n(z)=\frac{1}{2^n.2\pi i}\int_{C_0}\frac{(t^2-1)^n}{(t-z)^{n+1}}dt \tag{31}$$

This is Schilafli's integral for Legendre's polynomial when $n$ is an integer.

For non-integral values of $n$, the solution of Legendre equation can also be represented in terms of integration.

Assume

$$w=\frac{1}{2^n.2\pi i}\int_C\frac{(t^2-1)^n}{(t-z)^{n+1}}dt \tag{32}$$

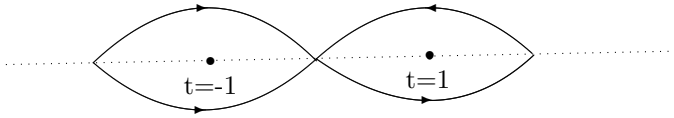


Figure 1: The contour $C_2$

where the contour $C$ is to be so chosen that this satisfies the Legendre differential equation

$$\frac{d}{dz}\left\{(1-z^2)\frac{dw}{dz}\right\}+n(n+1)w=0 \tag{33}$$

the left hand side of (33) by substituting (32) is

$$\frac{1}{2\pi i}\int_C \frac{(t^2-1)^n}{2^n}\left[\frac{d}{dz}\left\{\frac{(n+1)(1-z^2)}{(t-z)^{n+2}}\right\}+\frac{n(n+1)}{(t-z)^{n+1}}\right]dt$$

$$=\frac{(n+1)}{2^n.2\pi i}\int_C \frac{(t^2-1)^n}{(t-z)^{n+3}}\left[(n+2)(1-z^2)-2z(1-z)+n(t-z^2)\right]dt$$

$$=\frac{(n+1)}{2^n.2\pi i}\int_C \frac{(t^2-1)^n}{(t-z)^{n+3}}\left[2t(n+1)(1-z)-(n+2)(t^2-1)\right]dt$$

$$=\frac{(n+1)}{2^n.2\pi i}\int_C \frac{d}{dt}\left[\frac{(t^2-1)^{n+1}}{(t-z)^{n+2}}\right]dt$$

$$=\frac{(n+1)}{2^n.2\pi i}\left[\frac{(t^2-1)^{n+1}}{(t-z)^{n+2}}\right]_C \tag{34}$$

If $n$ be an integer then (34) vanishes when $C$ is any closed curve surrounding the point $t=z$ and thus (32) satisfies the differential equation (32) for such a contour.

If $n=\lambda$ (say) where $\lambda$ is not an integer then the contour $C$ is to be such that

$$\frac{(t^2-1)^{\lambda+1}}{(t-z)^{\lambda+2}}$$

returns to its original value after describing the contour $C$. The singularities of the above expression are at $t=\pm 1$ and at $t=z$.

If we consider a circle or any closed curve $C_1$ surrounding the singularities $t=z$ and $t=1$ in the clockwise sense the expression

$$\frac{(t^2-1)^{\lambda+1}}{(t-z)^{\lambda+2}}$$

is multiplied by

$$\frac{e^{-2\pi i(\lambda+1)}}{e^{-2\pi i(\lambda+2)}} = 1$$

when we go round the contour once.

Apart from $C_1$, there is another contour $C_2$ which is shown in Figure 10.1. $C_2$ consists of two loops one surrounding $t = -1$ in clockwise sense and the other surrounding $t = 1$ surrounding $t = 1$ in clockwise sense. On moving round this contour the expression is multiplied by $e^{-2\pi i(\lambda+1)}e^{2\pi i(\lambda+1)} = 1$ so that the contour $C_2$ also serves our purpose. The function with this contour integral form is Legendre formula of second kind.

## 2.11 Laplace Integral

Recall the Schlafli's integral representation (31) for Legendre polynomials

$$P_n(z) = \frac{1}{2^n.2\pi i}\int_{C_0}\frac{(t^2-1)^n}{(t-z)^{n+1}}dt$$

where $C$ is the closed contour enclosing the point $t = 1$ and $t = z$ but not $t = -1$.

Let us take for the contour $C$, the circle with the point $z$ as centre and of radius $|z^2-1|^{\frac{1}{2}}$. In order that this circle may enclosed the point $t = 1$, we must have $|z-1| < |z^2-1|^{\frac{1}{2}}$ i.e., $|z-1| < |z+1|$. Also, in order that this circle may not enclose the point $t = -1$, i.e., $t = -1$ be outside the circle, we must have $|z^2-1|^{\frac{1}{2}} < |z+1|$.

In order that this circle may enclosed of point $z = 1$ and may not $t = -1$ the point $z$ not lie to the right of the imaginary axis. Therefore

$$|\arg z| < \frac{\pi}{2}$$

If we take

$$t = z + (z^2-1)^{\frac{1}{2}}e^{i\phi}$$

be any point on the above circle with $z$ as centre and radius $\sqrt{z^2-1}$. Then $t$ describes $C$ once in the positive sense $\phi$ changes from $-\pi$ to $\pi$. Now,

$$t^2 - 1 = z^2 + (z^2-1)e^{2i\phi} + 2z(z^2-1)^{\frac{1}{2}}e^{i\phi} - 1$$

$$= (z^2-1)^{\frac{1}{2}}e^{2i\phi}\left[2z + (z^2-1)^{\frac{1}{2}}(e^{i\phi}+e^{-i\phi})\right]$$

$$= 2(z^2-1)^{\frac{1}{2}}e^{2i\phi}\left[z + (z^2-1)^{\frac{1}{2}}\cos\phi\right]$$

$$dt = (z^2-1)^{\frac{1}{2}}e^{i\phi}id\phi$$

Therefore

$$P_n(z) = \frac{1}{2^n.2\pi i}\int_{-\pi}^{\pi} \frac{2^n(z^2-1)^{\frac{n}{2}}e^{in\phi}\left\{z+(z^2-1)^{\frac{1}{2}}\cos\phi\right\}^n}{(z^2-1)^{n+1}e^{(n+1)i\phi}}(z^2-1)^{\frac{1}{2}}e^{i\phi}id\phi$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[z+(z^2-1)^{\frac{1}{2}}\cos\phi\right]^n d\phi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[z+(z^2-1)^{\frac{1}{2}}\cos\phi\right]^n d\phi \tag{35}$$

This is known as Laplace first integral.

# 3 Bessel Differential Equation

Bessel functions appear in a large class of physical problem. Bessel functions and closely related functions form a rich area of applied mathematics with many representations, many interesting and useful properties and many interrelations.

The Bessel differential equation of order $\lambda$ is

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2-\lambda^2)w = 0 \tag{36}$$

It is obvious that $z = 0$ is a regular singular point. To investigate the point at $\infty$ replacing the independent variable $z$ by $\frac{1}{t}$, the equation becomes

$$t^4\frac{d^2w}{dt^2} + t^3\frac{dw}{dt} + (1-\lambda^2t^2)w = 0 \tag{37}$$

This shows that the equation (40) has a regular singular point at $t = 0$ and hence the equation (36) has a singular point at $z = \infty$.

To obtain a series solution in the neighbourhood of $z = 0$ by Frobenius method, let us put

$$w = \sum_{k=0}^{\infty} c_k z^k \quad \text{in the equation (36).}$$

Then (36) becomes

$$\sum_{k=0}^{\infty} c_k(\rho+k)(\rho+k-1)z^{\rho+k} + \sum_{k=0}^{\infty} c_k(\rho+k)z^{\rho+k} + \sum_{k=0}^{\infty} c_k z^{\rho+k+2} - \lambda^2\sum_{k=0}^{\infty} c_k z^{\rho+k} = 0$$

The indicial equation is obtained by equating the coefficient of $z^\rho$ to zero as

$$\rho(\rho-1) + \rho - \lambda^2 = 0$$

That is, $\rho = \pm\lambda$

Now the coefficient of $z^{\rho+k}$ is

$$c_k\left[(\rho+k)^2-\lambda^2\right]+c_{k-2}=0$$

$$\text{i.e., } c_k = -\frac{c_{k-2}}{(\rho+\lambda+k)(\rho-\lambda+k)}$$

Coefficient of $z^{\rho+1}$ is $c_1\left[(\rho+1)^2-\lambda^2\right]=0$

But $[(\rho+1)^2-\lambda^2]\neq 0$ for $\rho=\pm\lambda$. Thus $c_1=0$.

This leads to the relation $c_3=c_5=c_7=\cdots=0$, i.e., the coefficient of all odd indices are zero and

$$c_{2k} = -\frac{c_{2k-2}}{(\rho+\lambda+2k)(\rho-\lambda+2k)},\ k=1,2,\ldots$$

**Case I:** $\rho=\lambda$. Without loss of generality we assume that $\lambda\geq 0$.

Therefore,

$$c_{2k} = -\frac{c_{2k-2}}{2k(2\lambda+2k)} = -\frac{1}{2^2}\frac{c_{2k-2}}{k(\lambda+k)}$$

$$\text{Hence } c_2 = -\frac{1}{2^2 1!}\frac{1}{(\lambda+1)}c_0$$

$$c_4 = (-1)^2\frac{1}{2^4 2!}\frac{1}{(\lambda+1)(\lambda+2)}c_0$$

$$c_6 = (-1)^3\frac{1}{2^6 3!}\frac{1}{(\lambda+1)(\lambda+2)(\lambda+3)}c_0 \text{ and so on.}$$

$$\text{Hence } w = z^\lambda[c_0+c_2z^2+c_4z^4+c_6z^6+\cdots]$$

$$= c_0z^\lambda\left[1-\frac{z^2}{2^2(\lambda+1)}+\frac{z^4}{2^4 2!(\lambda+1)(\lambda+2)}-\frac{z^6}{2^6.3!(\lambda+1)(\lambda+2)(\lambda+3)}+\cdots\right]$$

$$= c_0z^\lambda\left[1-\frac{(z/2)^2}{(\lambda+1)}+\frac{(z/2)^4}{2!(\lambda+1)(\lambda+2)}-\frac{(z/2)^6}{3!(\lambda+1)(\lambda+2)(\lambda+3)}+\cdots\right]$$

With a special choice $c_0=\dfrac{1}{2^\lambda\Gamma(\lambda+1)}$, $w$ is given by

$$w = (z/2)^\lambda\frac{1}{\Gamma(\lambda+1)}\left[1-\frac{(z/2)^2}{(\lambda+1)}+\frac{(z/2)^4}{2!(\lambda+1)(\lambda+2)}-\frac{(z/2)^6}{3!(\lambda+1)(\lambda+2)(\lambda+3)}+\cdots\right]$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{\lambda+2k}}{\Gamma(k+1)\Gamma(\lambda+k+1)}$$

This is called Bessel function (of first kind) of order $\lambda$ and is denoted by $J_\lambda(z)$.

Thus

$$J_\lambda(z) = \sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{\lambda+2k}}{\Gamma(k+1)\Gamma(\lambda+k+1)} \tag{38}$$

This is a convergent series for all values of $\lambda \geq 0$ and for all values of $z$.

**Case II:** $\lambda - (-\lambda) = 2\lambda$ is not an integer.

In this case the other independent solution is obtained by putting $-\lambda$ for $\lambda$. Hence

$$J_{-\lambda}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k-\lambda}}{\Gamma(k+1)\Gamma(-\lambda+k+1)} \tag{39}$$

This series is also convergent for all values of $z$.

It can be verified that both $J_\lambda(z)$ and $J_{-\lambda}(z)$ are solution of (36). If $2\lambda$ is not an integer then $J_\lambda(z)$ and $J_{-\lambda}(z)$ are two independent solution of (36) and the general solution of Bessel equation is

$$w = AJ_\lambda(z) + BJ_{-\lambda}(z).$$

**Case III:** $\lambda = 0$

In this case $J_\lambda(z)$ and $J_{-\lambda}(z)$ are not independent. One solution is $J_0(z)$, where

$$J_0(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{\Gamma(k+1)\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{(k!)^2}.$$

To find the second solution let

$$w = \sum_{k=0}^{\infty} c_k z^{\rho+k}$$

Proceeding in the similar way as in Case I, we have $c_{2k+1} = 0,\ k = 0, 1, 2, \ldots$ and

$$c_{2k} = -\frac{c_{2k-2}}{(\rho+\lambda+2k)(\rho-\lambda+2k)} = -\frac{c_{2k-2}}{(\rho+2k)^2},\ (\text{as } \lambda = 0)\ k = 1, 2, \ldots$$

Therefore

$$w = z^\rho [c_0 + c_2 z^2 + c_4 z^4 + c_6 z^6 + \cdots]$$

$$= c_0 z^\rho \left[1 + \frac{c_2}{c_0} z^2 + \frac{c_4}{c_2}\frac{c_2}{c_0} z^4 + \cdots\right]$$

$$= c_0 z^\rho \left[1 - \frac{z^2}{(\rho+2)^2} + \frac{z^4}{\{(\rho+2)(\rho+4)\}^2} - \frac{z^4}{\{(\rho+2)(\rho+4)(\rho+6)\}^2} + \cdots\right]$$

$$= c_0 z^\rho \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{\{(\rho+2)(\rho+4)\ldots(\rho+2k)\}^2}$$

$$= c_0 z^\rho \sum_{k=0}^{\infty} (-1)^k z^{2k} F(\rho)$$

where

$$F(\rho) = \frac{1}{\{(\rho+2)(\rho+4)\dots(\rho+2k)\}^2}$$

Now, $\log F(\rho) = -2\{\log(\rho+2) + \log(\rho+4) + \cdots + \log(\rho+2k)\}$

Differentiating with respect to $\rho$ we get

$$\frac{1}{F(\rho)}F'(\rho) = -\left[\frac{1}{\rho+2} + \frac{1}{\rho+4} + \cdots + \frac{1}{\rho+2k}\right]$$

That is,

$$F'(\rho) = -2F(\rho)\sum_{s=1}^{k}\frac{1}{\rho+2s}$$

This gives

$$F'(0) = -F(0)\sum_{s=1}^{k}\frac{1}{s} = -\frac{1}{(2.4.\dots.2k)^2}\sum_{s=1}^{k}\frac{1}{s}$$

$$= -\frac{1}{2^{2k}(k!)^2}\sum_{s=1}^{k}\frac{1}{s}$$

Hence

$$\frac{\partial w}{\partial \rho} = c_0 z^\rho \log z \sum_{k=0}^{\infty}(-1)^k z^{2k}F(\rho) + c_0 z^\rho \log z \sum_{k=0}^{\infty}(-1)^k z^{2k}F'(\rho)$$

Therefore the second solution denoted by $Y_0(z)$ is given by

$$Y_0(z) = \left(\frac{\partial w}{\partial \rho}\right)_{\rho=0} = c_0 \log z \sum_{k=0}^{\infty}(-1)^k z^{2k}F(\rho) + c_0\sum_{k=0}^{\infty}(-1)^k z^{2k}F'(\rho)$$

$$= \log z\left(c_0\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k}}{2^{2k}(k!)^2}\right) - c_0\sum_{k=0}^{\infty}(-1)^k z^{2k}\frac{1}{2^{2k}(k!)^2}\sum_{s=1}^{k}\frac{1}{s}$$

As $c_0 = 1$ because $\lambda = 0$. Therefore

$$Y_0(z) = J_0(z)\log z - \sum_{k=1}^{\infty}\frac{(-1)^k(z/2)^{2k}}{(k!)^2}\sum_{s=1}^{k}\frac{1}{s} \tag{40}$$

The function $Y_0(z)$ is called Neumann's function of the second kind of order zero. The function

$$\overline{Y_0(z)} = \frac{2}{\pi}[Y_0(z) - (\log 2 - \gamma)J_0(z)]$$

where $\gamma$ is Euler's constant, being obtained by adding to $Y_0(z)$ a function which is a constant multiple of $J_0(z)$, will also satisfy the Bessel equation. It is called Weber's Bessel function of the second kind of order zero.

Therefore

$$\overline{Y_0(z)} = \frac{2}{\pi}\{\log(z/2) + \gamma\}J_0(z) - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k(z/2)^{2k}}{(k!)^2}\sum_{s=1}^{k}\frac{1}{s}$$

The general solution of Bessel equation when $\lambda = 0$ is

$$w = AJ_0(z) + BY_0(z) \quad \text{or} \quad w = AJ_0(z) + B\overline{Y_0(z)}$$

**Case IV:** $2\lambda$ is an integer.

In this case the general solution of the Bessel equation can be written as

$$w = AJ_\lambda(z) + B\overline{Y_\lambda(z)}$$

where

$$\overline{Y_\lambda(z)} = \frac{1}{\pi}\{\log(z/2) + \gamma\}J_\lambda(z) - \frac{1}{\pi}\sum_{k=0}^{\lambda-1}\frac{(\lambda-k-1)!}{k!}\left(\frac{2}{z}\right)^{\lambda-2k}$$

$$-\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{\lambda+2k}}{k!(\lambda+k)!}\left\{\sum_{s=1}^{\lambda+k}\frac{1}{s} + \sum_{s=1}^{k}\frac{1}{s}\right\}$$

The function $\overline{Y_\lambda(z)}$ is called Weber's Bessel function of the second kind of order $\lambda$.

The Neumann Bessel function of second kind of order $\lambda$ is

$$Y_\lambda(z) = J_0(z)\log z - \frac{1}{\pi}\sum_{k=0}^{\lambda-1}\frac{(\lambda-k-1)!}{k!}\left(\frac{2}{z}\right)^{\lambda-2k}$$

$$-\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{\lambda+2k}}{k!(\lambda+k)!}\left\{\sum_{s=1}^{\lambda+k}\frac{1}{s} + \sum_{s=1}^{k}\frac{1}{s}\right\}$$

The equation (39) is also valid for integer values of $\lambda$. Let $\lambda = n$ where $n$ is a positive integer. Then

$$J_{-n}(z) = \sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{2k-n}}{k!(k-n)!}$$

Since $n$ is an integer, $\dfrac{1}{(k-n)!} = \dfrac{1}{\Gamma(k-n+1)} \to 0$ for $k = 0, 1, \ldots, n-1$. Hence the series may be considered to start with $k = n$. Replacing $k$ by $k+n$, we obtain

$$J_{-n}(z) = \sum_{k=0}^{\infty}\frac{(-1)^{k+n}(z/2)^{2k+n}}{k!(k+n)!}$$

**Theorem 2** *The Bessel functions $J_n(z)$ and $J_{-n}(z)$ are not independent, but they are related by*

$$J_n(z) = (-1)^n J_{-n}(z) \quad \textit{for integral } n.$$

**Proof.**

$$J_{-n}(z) = \sum_{k=0}^{\infty} \frac{(-1)^{k+n}(z/2)^{2k+n}}{k!(k+n)!} = (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+n}}{k!(k+n)!} = (-1)^n J_n(z)$$

### 3.0.1 Generating Function

The generating function has the advantage of focusing on the functions themselves rather than on the differential equations they satisfy. The generating function of Bessel differential equation is $g(z,t) = e^{\frac{z}{2}(t-\frac{1}{t})}$. Thus when $\lambda$ is a positive integer $J_\lambda(z)$ is equal to the coefficient of $t^\lambda$ in the expansion of $e^{\frac{z}{2}(t-\frac{1}{t})}$, i.e.,

$$e^{\frac{z}{2}(t-\frac{1}{t})} = \sum_{-\infty}^{\infty} t^\lambda J_\lambda(z)$$

**Proof.** $e^{\frac{z}{2}(t-\frac{1}{t})} = e^{\frac{zt}{2}} e^{-\frac{z}{2t}}$

$$= \left(1 + \frac{zt}{2} + \frac{1}{2!}\frac{z^2t^2}{2^2} + \cdots + \frac{1}{\lambda!}\frac{z^\lambda t^\lambda}{2^\lambda} + \frac{1}{(\lambda+1)!}\frac{z^{(\lambda+1)}t^{(\lambda+1)}}{2^{(\lambda+1)}} + \cdots\right)$$

$$\times \left(1 - \frac{z}{2t} + \frac{1}{2!}\frac{z^2}{2^2t^2} + \cdots + \frac{(-1)^\lambda}{\lambda!}\frac{z^\lambda}{2^\lambda t^\lambda} + \frac{(-1)^{\lambda+1}}{(\lambda+1)!}\frac{z^{(\lambda+1)}}{2^{(\lambda+1)}t^{(\lambda+1)}} + \cdots\right)$$

Coefficient of $t^\lambda$ in the product is

$$\frac{z^\lambda}{2^\lambda.\lambda!} - \frac{z^{\lambda+2}}{2^{\lambda+2}.(\lambda+1)!} + \frac{z^{\lambda+4}}{2!2^{\lambda+4}.(\lambda+2)!} - \cdots$$
$$= \frac{z^\lambda}{2^\lambda.\lambda!}\left[1 - \frac{1}{\lambda+1}\left(\frac{z}{2}\right)^2 + \frac{1}{2!(\lambda+1)(\lambda+2)}\left(\frac{z}{2}\right)^4 - \cdots\right]$$
$$= J_\lambda(z)$$

Coefficient of $t^{-\lambda}$ in the product is

$$\frac{(-1)^\lambda z^\lambda}{2^\lambda.\lambda!} - \frac{(-1)^{\lambda+1}z^{\lambda+2}}{2^{\lambda+2}.(\lambda+1)!} + \frac{(-1)^{\lambda+2}z^{\lambda+4}}{2!2^{\lambda+4}.(\lambda+2)!} + \cdots$$
$$= \frac{(-1)^\lambda z^\lambda}{2^\lambda.\lambda!}\left[1 - \frac{1}{\lambda+1}\left(\frac{z}{2}\right)^2 + \frac{1}{2!(\lambda+1)(\lambda+2)}\left(\frac{z}{2}\right)^4 - \cdots\right]$$
$$= (-1)^\lambda J_\lambda(z) = J_{-\lambda}(z)$$

The following result can easily be deduced using the concept of generating function rather than using the differential equation.

**Example 3.1** Show that

$$\cos(z\sin\theta) = J_0(z) + 2\sum_{n=1}^{\infty} J_{2n}(z)\cos 2n\theta \tag{41}$$

$$\sin(z\sin\theta) = 2\sum_{n=0}^{\infty} J_{2n+1}(z)\sin(2n+1)\theta \tag{42}$$

What will be the cases when $\theta = \frac{\pi}{2}$?

**Solution.** From generating function

$$\begin{aligned} e^{\frac{z}{2}(t-\frac{1}{t})} &= \sum_{n=-\infty}^{\infty} t^n J_n(z) \\ &= (J_0 + J_1 t + J_2 t^2 + \cdots) + \{(-1)^1 J_1 t^{-1} + (-1)^2 J_2 t^{-2} + (-1)^3 J_3 t^{-3} + \cdots\} \\ &= J_0 + J_1(t - t^{-1}) + J_2(t^2 + t^{-2}) + J_3(t^3 + t^{-3}) + \cdots \end{aligned}$$

Let $t = \cos\theta + i\sin\theta$.

Then $t^n = \cos n\theta + i\sin n\theta$ and $t^{-n} = \cos n\theta - i\sin n\theta$.

Therefore $t^n + t^{-n} = 2\cos n\theta$ and $t^n - t^{-n} = 2i\sin n\theta$.

Substituting these values to the above relation we get

$$e^{iz\sin\theta} = J_0 + 2iJ_1\sin\theta + 2J_2\cos 2\theta + 2iJ_3\sin 3\theta + 2J_4\sin 4\theta + \cdots$$

Equating the real and imaginary parts from both sides we have

$$\cos(z\sin\theta) = J_0(z) + 2\sum_{n=1}^{\infty} J_{2n}(z)\cos 2n\theta$$

and

$$\sin(z\sin\theta) = 2\sum_{n=0}^{\infty} J_{2n+1}(z)\sin(2n+1)\theta$$

**Second part:** when $\theta = \frac{\pi}{2}$ then above relations become

$$\cos z = J_0(z) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(z)$$

and

$$\sin z = 2\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(z)$$

This example suggest that $\sin z$ and $\cos z$ can be expressed in terms of Bessel functions.

## 3.1 Recurrence Relation of $J_\lambda(z)$

Different recurrence relations are available for Bessel functions which are most important to many applications. These are presented below.

Differentiating partially the relation $e^{\frac{z}{2}(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} t^\lambda J_\lambda(z)$ with respect to $z$, we obtain

$$\frac{1}{2}\left(t-\frac{1}{t}\right)e^{\frac{z}{2}(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} t^\lambda J'_\lambda(z)$$

$$\text{or} \quad \left(t-\frac{1}{t}\right)\sum_{n=-\infty}^{\infty} t^\lambda J_\lambda(z) = 2\sum_{n=-\infty}^{\infty} t^\lambda J'_\lambda(z)$$

Equating the coefficient of $t^\lambda$, we get

$$2J_\lambda(z) = J_{\lambda-1}(z) - J_{\lambda+1}(z) \tag{43}$$

Again differentiating partially the relation

$$e^{\frac{z}{2}(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} t^\lambda J_\lambda(z)$$

with respect to $t$ we obtain

$$\frac{z}{2}\left(1+\frac{1}{t^2}\right)e^{\frac{z}{2}(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} \lambda t^{\lambda-1} J_\lambda(z)$$

This gives,

$$z\left(1+\frac{1}{t^2}\right)\sum_{n=-\infty}^{\infty} t^\lambda J_\lambda(z) = 2\sum_{n=-\infty}^{\infty} \lambda t^{\lambda-1} J_\lambda(z)$$

Equating the coefficient of $t^{\lambda-1}$ from both sides we obtain

$$\frac{2\lambda}{z}J_\lambda(z) = J_{\lambda-1}(z) + J_{\lambda+1}(z) \tag{44}$$

Adding (43) and (44) we get

$$2J'_\lambda(z) + \frac{2\lambda}{z}J_\lambda(z) = 2J_{\lambda-1}(z)$$

This relation can be written as

$$zJ'_\lambda(z) = zJ_{\lambda-1}(z) - \lambda J_\lambda(z) \tag{45}$$

Subtracting (43) from (44) we get

$$\frac{2\lambda}{z}J_\lambda(z) - 2J'_\lambda(z) = 2J_{\lambda+1}(z)$$

That is,

$$zJ'_\lambda(z) = \lambda J_\lambda(z) - zJ_{\lambda+1}(z) \tag{46}$$

Adding (45) and (46) we obtain another recurrence relation as

$$J_{\lambda-1}(z) - J_{\lambda+1}(z) = 2J'_\lambda(z) \tag{47}$$

**Example 3.2** Prove that

$$\frac{d}{dz}\left[z^\lambda J_\lambda(z)\right] = z^\lambda J_{\lambda-1}(z)$$

$$\frac{d}{dz}\left[z^{-\lambda} J_\lambda(z)\right] = -z^{-\lambda} J_{\lambda+1}(z)$$

**Solution.** Multiplying both sides of (45) by $z^{\lambda-1}$ we obtain

$$z^\lambda J'_\lambda(z) = z^\lambda J_{\lambda-1}(z) - \lambda z^{\lambda-1} J_\lambda(z)$$

$$\text{or} \quad z^\lambda J'_\lambda(z) + \lambda z^{\lambda-1} J_\lambda(z) = z^\lambda J_{\lambda-1}(z)$$

$$\text{or} \quad \frac{d}{dz}\left[z^\lambda J_\lambda(z)\right] = z^\lambda J_{\lambda-1}(z)$$

Multiplying both sides of (46) by $z^{-\lambda-1}$ we obtain

$$z^{-\lambda} J'_\lambda(z) = \lambda z^{-\lambda-1} J_\lambda(z) - z^{-\lambda} J_{\lambda+1}(z)$$

$$\text{or} \quad z^{-\lambda} J'_\lambda(z) - \lambda z^{-\lambda-1} J_\lambda(z) = -z^{-\lambda} J_{\lambda+1}(z)$$

$$\text{or} \quad \frac{d}{dz}\left[z^{-\lambda} J_\lambda(z)\right] = -z^{-\lambda} J_{\lambda+1}(z)$$

**Example 3.3** Prove that

(i) $J'_0(z) = -J_1(z)$

(ii) $J_2(z) = J''_0(z) - \frac{1}{z} J'_0(z)$

**Solution.** The recurrence relation (46) is

$$zJ'_\lambda(z) = \lambda J_\lambda(z) - zJ_{\lambda+1}(z)$$

Putting $\lambda = 0$ we have

$$zJ'_0(z) = -zJ_1(z)$$

$$\text{i.e.,} \quad J'_0(z) = -J_1(z)$$

Differentiating w. r. t. $z$, we get

$$J'_1(z) = -J''_0(z) \tag{48}$$

Putting $\lambda = 0$ in (46) we have

$$zJ'_1(z) = J_1(z) - zJ_2(z)$$

$$\text{or} \quad z[-J''_0(z)] = -J'_0(z) - zJ_2(z) \quad [\text{Using (48)}]$$

$$\text{or} \quad J_2(z) = J''_0(z) - \frac{1}{z} J'_0(z)$$

## 3.2 Bessel Differential Equation from Recurrence Relations

Suppose we consider a set of functions $w_\lambda(z)$ which satisfies the recurrence relation (45), but $w_\lambda(z)$ not necessarily given by the series (38). The equation (45) can be written as

$$zw_\lambda(z) = zw_{\lambda-1}(z) - \lambda w_\lambda(z) \tag{49}$$

Differentiating w. r. t. $z$ we have

$$zw''_\lambda(z) + (\lambda+1)w'_\lambda(z) - zw'_{\lambda-1}(z) - w_{\lambda-1}(z) = 0 \tag{50}$$

Multiplying this equation by $z$ and then subtracting (48) multiplied by $\lambda$, we get

$$z^2w''_\lambda(z) + zw'_\lambda(z) - \lambda^2 w_\lambda(z) + (\lambda-1)zw_{\lambda-1}(z) - z^2w'_{\lambda-1}(z) = 0 \tag{51}$$

Using the equation (46) and replacing $\lambda$ by $\lambda - 1$ we have

$$zw'_{\lambda-1} = (\lambda-1)w_{\lambda-1} - zw_\lambda$$

Substituting the value of $zw'_{\lambda-1}(z)$ to the last term of (51) we get

$$z^2w''_\lambda(z) + zw'_\lambda(z) - \lambda^2 w_\lambda(z) + (\lambda-1)zw_{\lambda-1}(z) - z\{(\lambda-1)w_{\lambda-1} - zw_\lambda\} = 0$$

On simplification this equation becomes

$$z^2w''_\lambda(z) + zw'_\lambda(z) + (z^2 - \lambda^2)w_\lambda(z) = 0$$

This is well known Bessel equation. Hence any functions, $w_\lambda(z)$ that satisfy the recurrence relations (45) and (46), satisfy Bessel differential equation, that is the unknown function $w_\lambda(z)$ is Bessel function.

## 3.3 Representation of Bessel Functions as Continued Fraction

The Bessel functions can be expressed as continued fraction. The equation (44) can be written as

$$J_{\lambda-1} + J_{\lambda+1} = \frac{2\lambda}{z}J_\lambda$$

That is

$$\frac{J_{\lambda-1}}{J_\lambda} = \frac{2\lambda}{z} - \frac{J_{\lambda+1}}{J_\lambda} = \frac{2\lambda}{z} - \frac{1}{\frac{J_\lambda}{J_{\lambda+1}}}$$

$$= \frac{2\lambda}{z} - \cfrac{1}{\frac{2(\lambda+1)}{z} - \cfrac{1}{\frac{2(\lambda+2)}{z} - \cfrac{1}{\frac{2(\lambda+3)}{z} - \cdots}}} \tag{52}$$

Multiplying (52) by $z$ we get

$$z\frac{J_{\lambda-1}}{J_\lambda} = 2\lambda - \frac{z^2}{2(\lambda+1) - \frac{z^2}{2(\lambda+2) - \frac{z^2}{2(\lambda+3)-\cdots}}} \tag{53}$$

In particular when $\lambda = 0, \frac{1}{2}$ we have

$$z\frac{J_{-1}}{J_0} = -\frac{z^2}{2 - \frac{z^2}{4 - \frac{z^2}{6-\cdots}}}$$

$$z\frac{J_{-\frac{1}{2}}}{J_{\frac{1}{2}}} = 1 - \frac{z^2}{3 - \frac{z^2}{5 - \frac{z^2}{7-\cdots}}} \tag{54}$$

**Example 3.4** Show that

$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\sin z$$
$$J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\cos z$$
$$z\cot z = 1 - \frac{z^2}{3 - \frac{z^2}{5 - \frac{z^2}{7-\cdots}}}$$
$$\{J_{\frac{1}{2}}(z)\}^2 + \{J_{-\frac{1}{2}}(z)\}^2 = \frac{2}{\pi z}$$

**Solution.** We have

$$\begin{aligned} J_{\frac{1}{2}}(z) &= \sum_0^\infty \frac{(-1)^k (z/2)^{\frac{1}{2}+2k}}{\Gamma(k+1)\Gamma(\frac{1}{2}+k+1)} \\ &= \sqrt{\frac{2}{z}} \sum_0^\infty \frac{(-1)^k (z/2)^{1+2k}}{\Gamma(k+1)\Gamma(\frac{3}{2}+k)} \\ &= \sqrt{\frac{2}{z}} \frac{1}{\Gamma(1/2)} \left[\frac{\frac{z}{2}}{\frac{1}{2}} - \frac{(\frac{z}{2})^3}{1!\frac{3}{2}\frac{1}{2}} + \frac{(\frac{z}{2})^5}{2!\frac{5}{2}\frac{3}{2}\frac{1}{2}} - \cdots\right] \\ &= \sqrt{\frac{2}{\pi z}}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right] = \sqrt{\frac{2}{z}}\sin z \end{aligned} \tag{55}$$

Similarly,

$$J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{z}}\cos z \tag{56}$$

Dividing (58) by (55) we obtain

$$\frac{J_{-\frac{1}{2}}(z)}{J_{\frac{1}{2}}(z)} = \cot z \tag{57}$$

Combination of the results (59) and (54) give

$$z\cot z = 1 - \cfrac{z^2}{3 - \cfrac{z^2}{5 - \cfrac{z^2}{7 - \cdots}}}$$

Squaring and adding (55) and (58) we get

$$\{J_{\frac{1}{2}}(z)\}^2 + \{J_{-\frac{1}{2}}(z)\}^2 = \frac{2}{\pi z}$$

## 3.4 Integral representation

Recall the equations (41) and (42) as

$$\cos(z\sin\theta) = J_0(z) + 2\sum_{n=1}^{\infty} J_{2n}(z)\cos 2n\theta$$

$$\sin(z\sin\theta) = 2\sum_{n=0}^{\infty} J_{2n+1}(z)\sin(2n+1)\theta$$

By employing the orthogonality properties of cosine and sine functions

$$\int_0^{\pi} \cos n\theta \cos m\theta d\theta = \frac{\pi}{2}\delta_n^m$$

$$\int_0^{\pi} \sin n\theta \sin m\theta d\theta = \frac{\pi}{2}\delta_n^m$$

in which $m$ and $n$ are non negative integers. Now,

Now, $\quad \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \cos(z\sin\theta)\cos n\theta d\theta$

$$= \frac{1}{\pi}\int_0^{\pi}\left[J_0(z) + 2\sum_{m=1}^{\infty} J_{2m}(z)\cos 2m\theta\right]\cos n\theta d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi} J_0(z)\cos n\theta d\theta + \frac{2}{\pi}\sum_{m=1}^{\infty} J_{2m}(z)\int_0^{\pi}\cos 2m\theta\cos n\theta d\theta$$

$$= \frac{1}{\pi}J_0(z)\left[\frac{\sin n\theta}{n}\right]_0^{\pi} + \frac{2}{\pi}\sum_{m=1}^{\infty} J_{2m}(z)\frac{\pi}{2}\delta_n^{2m}$$

$$= \begin{cases} J_n(z), & \text{when } n \text{ is even} \\ 0, & \text{when } n \text{ is odd} \end{cases} \qquad \text{Since } \delta_n^{2m} = 1 \text{ when } m = \frac{n}{2}, \text{ otherwise } \delta_n^{2m} = 0$$

Similarly,

$$\frac{1}{\pi}\int_0^{\pi}\sin(z\sin\theta)\sin n\theta d\theta = \begin{cases} 0, & \text{when } n \text{ is even} \\ J_n(z), & \text{when } n \text{ is odd} \end{cases}$$

Addition of these two relatives gives

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi}\left[\cos(z\sin\theta)\cos n\theta + \sin(z\sin\theta)\sin n\theta\right]d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi}\cos(n\theta - z\sin\theta)d\theta \quad n = 0, 1, 2, \ldots$$

As a special case,

$$J_0(z) = \frac{1}{\pi}\int_0^{\pi}\cos(z\sin\theta)d\theta \tag{58}$$

Note that $\cos(z\sin\theta)$ repeats itself in all four quadrants ($\theta_1 = \theta, \pi-\theta, \pi+\theta, -\theta$). The equation (58) can be written as

$$J_0(z) = \frac{1}{2\pi}\int_0^{2\pi}\cos(z\sin\theta)d\theta \tag{59}$$

On the other hand, $\sin(z\sin\theta)$ reverses its sign in the third and fourth quadrants so that

$$\frac{1}{2\pi}\int_0^{2\pi}\cos(z\sin\theta)d\theta = 0 \tag{60}$$

Adding (59) and $i$ times (60), we obtain

$$J_0(z) = \frac{1}{2\pi}\int_0^{2\pi}e^{iz\sin\theta}d\theta = \frac{1}{2\pi}\int_0^{2\pi}e^{iz\cos\theta}d\theta \tag{61}$$

The Bessel function can be expressed in terms of contour integration as follows.

**Theorem 3** *For a suitable contour $C$ enclosing $t = 0$, the Bessel function of order $\lambda$ is given by*

$$J_\lambda = \frac{1}{2\pi i}\int_C e^{\frac{z}{2}(t-\frac{1}{t})}t^{\lambda+1}dt$$

**Proof:** The Laurent series for an analytic function $f(t)$ is

$$f(t) = \sum_{n=-\infty}^{\infty} a_n t^n \quad \text{where} \quad a_n = \frac{1}{2\pi i}\int_C \frac{f(t)}{t^{n+1}}dt \tag{62}$$

Let $f(t) = e^{\frac{z}{2}(t-\frac{1}{t})}$. Again

$$e^{\frac{z}{2}(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} t^{\lambda} J_{\lambda}(z) \tag{63}$$

For this $f(t)$, (62) becomes

$$e^{\frac{z}{2}(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} a_{\lambda} t^n \text{where} \quad a_{\lambda} = \frac{1}{2\pi i}\int_C \frac{f(t)}{t^{\lambda+1}}dt \tag{64}$$

Comparing (63) and (64) we have

$$J_{\lambda}(z) = a_{\lambda} = \frac{1}{2\pi i}\int_C \frac{e^{\frac{z}{2}(t-\frac{1}{t})}}{t^{\lambda+1}}dt$$

The contour $C$ is shown in Figure
The contour $C$ consists of
1. $-\infty$ to $-1$ along the real axis,
2. the unit circle in the anticlockwise sense
3. the real axis from $-1$ to $-\infty$ in the opposite direction

**When $\lambda$ is not an integer**

In this case, $t^{-(\lambda+1)}$ has branch point at $t = 0$ and $\pm\infty$ and a cut is necessary along the negative real axis from 0 to $-\infty$ and thus we get

$$J_{\lambda}(z) = a_{\lambda} = \frac{1}{2\pi i}\int_{C_1} \frac{e^{\frac{z}{2}(t-\frac{1}{t})}}{t^{\lambda+1}}dt$$

**When $\lambda$ is not an integer**

The contour $C$ becomes a unit circle. Since the two integrals along the negative real axis cancel each other and finally

$$J_{\lambda}(z) = a_{\lambda} = \frac{1}{2\pi i}\int_{C_0} \frac{e^{\frac{z}{2}(t-\frac{1}{t})}}{t^{\lambda+1}}dt$$

where $C_0$ is a unit circle containing $t = 0$.

**Example 3.5** For an integral value of $n$, show that

$$J_n(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{i(z\sin\theta - n\theta)}d\theta$$

**Solution.** We have

$$J_n(z) = \frac{1}{2\pi i}\int_C \frac{e^{\frac{z}{2}(t-\frac{1}{t})}}{t^{n+1}}dt$$

where $C$ is the contour represents a unit circle with centre at the origin.

Let $t = \cos\theta + i\sin\theta = e^{i\theta}$

Then $t - \frac{1}{t} = 2i\sin\theta$ and $\theta$ varies from 0 to $2\pi$.

Therefore,

$$J_n(z) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{\frac{z}{2}(2i\sin\theta)}}{(e^{i\theta})^{n+1}}.ie^{i\theta}d\theta = \frac{1}{2\pi}\frac{1}{2\pi}\int_0^{2\pi} e^{i(z\sin\theta - n\theta)}d\theta$$

## *EXERCISES*

(i) Express the following expressions in terms of Legendre polynomials

(a) $3x - 2x^2 + x^3$

(b) $x^4 - 8x^2 - 1$

(c) $1 + x + x^2$

(d) $\sin x$

(ii) Expand the following function in terms of Legendre polynomials

$$f(x) = \begin{cases} 0, & if \ -1 \le x < 0 \\ 1, & if \ 0 < x \le 1 \end{cases}.$$

(iii) Show that

$$1+3P_1(z)+5P_2(z)+7P_3(z)+\cdots+(2n+1)P_n(z) = \frac{d}{dz}\left[P_{n+1}(z) + P_n(z)\right]$$

(iv) Prove the following recurrence relations

(a) $P'_{n+1}(z) - zP'_n(z) = (n+1)P_n(z)$

(b) $(n+1)P_{n+1}(z) - (2n+1)zP_n(z) + nP_{n-1}(z) = 0$

(c) $zP'_n(z) - P'_{n-1}(z) = nP_n(z)$

(d) $P'_{n+1}(z) - zP'_{n-1}(z) = (2n+1)P_n(z)$

(e) $(1-z^2)P'_n(z) = nP_{n-1}(z) - nzP_n(z)$

(v) Show that $(1-2zt+t^2)^{-\frac{1}{2}}$ is the generating function of Legendre polynomials.

(vi) Show that $P_n(z)$ is the coefficient of $t^n$ in the expansion of $(1-2zt+t^2)^{-\frac{1}{2}}$.

(vii) Show that the Legendre differential equation can be written in the form

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{dP_n(\theta)}{d\theta}\right]+n(n+1)P_n(\cos\theta)=0$$

(viii) Show that

$$\int_0^{\pi} P_n(\cos\theta)P_n(\cos\theta)\sin\theta d\theta=\begin{cases}0, & if\ m\neq n\\ \dfrac{2}{2n+1}, & if\ m=n\end{cases}$$

(ix) Show that

$$\frac{1+t}{t\sqrt{1-2tz+t^2}}-\frac{1}{t}=\sum_{n=0}^{\infty}\left[P_n(z)+P_{n+1}(z)\right]t^n$$

(x) Prove that

$$\frac{1-t^2}{\{1-2tz+t^2\}^{\frac{3}{2}}}=\sum_{n=0}^{\infty}(2n+1)P_n(z)t^n$$

(xi) Show that

$$1+\sum_{n=0}^{\infty}\frac{1}{n+1}P_n(\cos\theta)=\log\left[\frac{1+\sin\theta/2}{\sin\theta/2}\right].$$

(xii) Show that

$$P_n(z)=\frac{d^n}{dz^n}\left[(1-z)^n\left\{1-\frac{1-z}{2}\right\}^n\right]$$

(xiii) Show that

$$\int_0^1 P_{2n}(z)P_{2n+1}(z)dz=\int_0^1 P_{2n}(z)P_{2n-1}(z)dz$$

(xiv) Prove that

(a) $\displaystyle\int_{-1}^{1}[P_n(z)]^2dz=\frac{2}{2n+1}$

(b) $\displaystyle\int_{-1}^{1}[P_n'(z)]^2dz=n(n+1)$

(c) $\int_{-1}^{1} zP_n(z)P_n'(z)dz = \dfrac{2n}{2n-1}$

(d) $\int_{-1}^{1} zP_n(z)P_{n-1}(z)dz = \dfrac{2n}{4n^2-1}$

(xv) Show that

(a) $\int_{-1}^{1} z^2P_n^2(z)dz = \dfrac{1}{8}\left[\dfrac{1}{2n-1} + \dfrac{6}{2n+1} + \dfrac{1}{2n+3}\right]$

(b) $\int_{-1}^{1} (z^2-1)P_{n+1}(z)P_n'(z)dz = \dfrac{2n(n+1)}{(2n+1)(2n+3)}$

(xvi) Prove that

$$P_n(\cos\theta) = \sum_{k=0}^{\infty}(-1)^k({}^nC_k)^2\cos^{2n-2}\theta\sin^{2k}\theta$$

(xvii) Show that

$$\int_0^{\pi} P_{2n}(\cos\theta)d\theta = \left[\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\right]^2$$

(xviii) Prove that

$$\int_{-1}^{1} z(1-z^2)P_n'(z)P_m'(z)dz = \begin{cases} 0, & unless \;\; m = n \pm 1 \\ \frac{2n(n^2-1)}{4n^2-1}\delta_{m,n-1}, & if \;\; m < n \\ \frac{2n(n+2)(n+1)}{(2n+1)(2n+3)}\delta_{m,n+1}, & if \;\; m > n \end{cases}$$

(xix) Operating in spherical polar co-ordinates, show that

$$\frac{\partial}{\partial z}\left[\frac{P_n(\cos\theta)}{r^{n+1}}\right] = -(n+1)\frac{P_{n+1}(\cos\theta)}{r^{n+2}}$$

(xx) Prove that

$$P_n'(1) = \frac{1}{2}n(n+1)$$

(xxi) Prove that, for $n \geq 1$,

$$\int_{-1}^{1} \log(1-z)P_n(z)dz = -\frac{2}{n(n+1)}$$

(xxii) Prove that

$$\int_{-1}^{1} P_n(z)dz = -\frac{1}{(2n+1)}\left[P_{n-1}(z) - P_{n+1}(z)\right]$$

(xxiii) Prove that

(a) $P_n(0) = 0$ for odd $n$ and

(b) $P_n(0) = \frac{(-1)^{\frac{n}{2}}\, n!}{2^n\{(n/2)!\}^2}$ for even $n$.

(xxiv) Prove that

(a) $\int_0^z z^n J_{n-1}(z)dz = z^n J_n(z)$

(b) $\int_0^z z^{n+1} J_n(z)dz = z^{n+1} J_{n+1}(z)$

(c) $\int_0^\infty \frac{J_n(z)}{z}dz = \frac{1}{n}$

(d) $\int_0^z J_0(z)dz = 1$

(xxv) Prove that (a) $4\int J_{n+1}(z)dz = \int J_{n-1}(z)dz - 2J_n(z)$

(b) $4\frac{d^2}{dz^2}[J_n(z)] = J_{n-2}(z) - 2J_n(z) + J_{n+2}(z)$

(c) $\int_0^\infty e^{-z} J_0(z)dz = \frac{1}{\sqrt{2}}$

(xxvi) Show that by the use of recurrence formulae that (a) $J_0'' = \frac{J_2 - J_0}{2}$

(b) $\frac{J_2(z)}{J_1(z)} = \frac{1}{z} - \frac{J_0''(z)}{J_0'(z)}$

(c) $J_0'''(z) + 3J_0'(z) + J_3(z) = 0$